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The evolution of a batch-immigration death process subject to counts

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A bivariate batch immigration-death process is developed to study the degree to which the fundamental structure of a hidden stochastic process can be inferred purely from counts of escaping individuals. This question is of immense importance in fields such as quantum optics, where externally based radiation elucidates the nature of the underlying electromagnetic radiation process. Batches of i immigrants enter the population at rate αq_i, and each individual dies independently at rate μ. General expressions are developed for the population size cumulants and probabilities, together with those for the associated counting process. The strong link between these two structures is highlighted through two specific examples, involving k-batch immigration for $i=k$, and Schoenberg-batch immigration over $i=2^m$ ($m=0, 1, 2, \ldots$), and shows that high quality inferences on the hidden population process can be inferred purely from externally counted observations.

Keywords: batch immigration; counts; hidden events; stochastic population process; quantum optics; Schoenberg distribution

1. Introduction

Complex systems lying far from equilibrium in a highly correlated state not only have the potential for exhibiting extremely rich dynamic behaviour, but the fluctuations in these systems are often large and are frequently described by scale-free probability density functions. A wide range of examples exhibiting such scale-free fluctuations that have been observed in discrete random phenomena (see Matthews *et al.* 2003) include systems as diverse as the World Wide Web (Barabási & Albert 1999), organic metabolisms (Jeong *et al.* 2000), protein interactions (Jeong *et al.* 2001) and social networks (Albert & Barabási 2000). The associated order distributions of such systems describe the number N of links that connect the nodes, and these are typically scale-free with probability density function (PDF) $p(N)\sim 1/N^{\nu}$. In the classic sandpile paradigm, the distance travelled by grains in an avalanche, called the 'flight length', raises a paradox since it has a power-law density, and so the variance of the distance travelled does not exist, thereby implying the need for infinite energy to transport the particles. Detailed analysis, however, reveals that individual flight

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lengths comprise a sum of steps where the length of each step does have finite variance, and it is because the number of steps has a power-law tail that causes $p(N) \sim 1/N^{\nu}$ (Hopcraft *et al.* 2001, 2002).

Not only are systems described by power-law distributions typified by large fluctuations, which raises the practical question of how their behaviour can be measured, but often observations can only be made on emission counts from the process and not from direct measurement of the population size itself. Thus, assuming that fluctuations in the internal population are at least partially transferred to an external series of countable events, we have to consider the general question of whether it is possible to determine the fundamental structure of a hidden stochastic population process purely from counts of escaping individuals (Gillespie & Renshaw submitted).

This visualization of a physical process in terms of a stochastic emigration population model (Jakeman & Shepherd 1984; Shepherd 1984) is of immense importance in fields such as quantum optics, where externally based radiation elucidates the nature of the underlying electromagnetic radiation process (Srinivasan 1988). A particularly interesting problem concerns the stochastic evolution of photon populations within optical cavities. This field developed rapidly following the inception of the laser (Shimoda *et al.* 1957), and understanding the counting statistics of photo-electron pulses registered by detectors such as photomultiplier tubes has been essential for the interpretation of experimental measurements (Saleh 1978). Properties of the number of emigrants leaving the population in a fixed time-interval correspond precisely to the experimentally measured photon-counting statistics, and so provide an indirect measure of the evolution of the cavity population itself. Moreover, as well as providing additional insight into the quantum formulation of the problem, the interchange of quantum models and techniques with those in the field of classical population processes deeply enriches both fields of study.

For many years it was generally accepted that the doubly stochastic Poisson process provided an adequate representation of photodetection. In this classical situation the probability of registering c pulses during a time-interval of fixed length T is given by a non-homogeneous Poisson process, whose intensity I takes the PDF $f(I)$ (see Mandel 1959; Cox & Lewis 1966). It follows that the variance to mean ratio, ρ, of c must be at least that of the Poisson distribution, namely one, and light that can be represented in this way is termed *classical*. Examples include coherent laser light (Poisson) and thermal light (geometric). However, not only is this representation unable to cope with sub-Poissonian models ($\rho < 1$) for photon-flux, neither can it support a wide range of super-Poissonian models ($\rho > 1$). Such light is called *non-classical* (Jakeman *et al.* 1995), and intense activity in the development of experimental methods for generating it has produced a range of techniques that provide overwhelming evidence for its existence. However, since most applications require consideration of both wave and particle properties, theoretical treatments have generally avoided a population process approach. Jakeman *et al.* (1995) consequently developed a paired-immigration death model, and although the equilibrium distribution is super-Poissonian, the implicit odd–even effects ensure that the resulting light is non-classical. The attraction of this simple model lies not only in its analytical tractability, but also in that it represents one of the earliest mechanisms used to produce non-classical light through parametric down-conversion in a non-linear crystal (Burnham & Weinberg 1970) and so is realisable experimentally.

The approach taken by Gillespie & Renshaw (submitted) is to generalize this multiple immigration death (MID) theme subject to the constraints of analytic tractability. Expressions are derived for the probabilities $p_{nc}(t)$ that the population contains n individuals at time $t>0$ and c emigrants have been counted under single, paired and triple immigration regimes. Not only do these probabilities, and their associated moments, show 'saw-tooth' behaviour (e.g. $p_n(t) > p_{n+1}(t) < p_{n+2}(t) > \dots$), but information on the marginal counting probabilities $p_c(t)$ (and moments) provides a surprising amount of insight into the population structure $\{p_n(t)\}$. In the triple immigration case, for example, both $p_n(t)$ and $p_c(t)$ exhibit 'triple saw-tooth' behaviour, in that their forms involve different combinations of Hermite polynomials depending on whether n and c equal $3m$, $3m+1$ or $3m+2$ for integer m. This close correspondence between population and counting statistics is of fundamental significance, since it means that if a stochastic process is developing within a hidden system, with the only information being provided by the event times of 'escaping individuals', then the properties of the hidden population process may be inferred purely from knowledge of the counting statistics. This feature remains even when births (BMID model) are introduced, and Jakeman *et al.* (2002) show that the BMID and MID population processes may only be distinguished through the behaviour of their high-order correlation properties. Now, it may be anticipated that these properties will be reflected not only in the higher-order correlation properties of the emigrants, but also in the higher-order moments of single-interval emigrant count fluctuations. This is because the single-interval integrated statistics reflect the population evolution over a finite sample time and, therefore, are strongly influenced by the associated correlations. For this reason, we develop a general theory of batch immigration counting processes by extending our earlier results based on paired and triple immigration, since these neatly illustrate the fundamental moment and distributional issues involved. First, we consider the probability and moment structure of the general linear immigration-death process in which batches of i immigrants arrive randomly at rate q_i. Next, we examine the situation exposed by Schoenberg (1983) in which the normally close relationship between probabilities and moments breaks down, in the sense that there exists an infinite set of probability distributions all of whose members possess the same moment structure.

2. The general batch-immigration death process

Given that Kolmogorov equations can be solved exactly only for a small array of stochastic processes, a full analytic treatment of our fundamental question (namely whether a hidden Markov structure can be determined purely from counts of escaping individuals) is clearly mathematically intractable. Moreover, since the utilization of stochastic approximation techniques is currently a 'black art', any attempt to develop a general analytic argument would result in a confounding between the level and type of approximation employed and the results obtained. Fortunately, several previous studies have shown that the immigration-death process is not only mathematically tractable, robust and ubiquitous in its areas of application, but also that inferences and insights gleaned from it often have strong relevance across a wide range of stochastic processes. It is therefore with considerable confidence that we use it here as a touchstone against which we can judge the general case.

Generalizing the specific and highly detailed double and triple immigration analysis developed by Gillespie & Renshaw (submitted), suppose that batches of i immigrants enter the population with constant rate αq_i, where $\sum_{i=1}^{\infty} q_i = 1$, and let each individual die at rate μ. Then the Kolmogorov forward equation for $p_n(t)$ takes the form

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \mu(n+1)p_{n+1}(t) + \alpha \sum_{i=1}^{\infty} q_i p_{n-i}(t) - (\mu n + \alpha)p_n(t), \tag{2.1}$$

where $p_{n-i}(t) \equiv 0$ for all $i > n$. Whence on denoting the associated probability generating function (PGF) by

$$Q(z;t) \equiv \sum_{n=0}^{\infty} z^n p_n(t), \tag{2.2}$$

on multiplying equation (2.1) by z^n and summing over n, we obtain

$$\frac{\partial Q}{\partial t} = \mu(1-z)\frac{\partial Q}{\partial z} + \alpha Q \sum_{i=1}^{\infty} q_i (z^i - 1). \tag{2.3}$$

On assuming that the initial population is of size zero at time $t=0$, this equation can be solved either by using Lagrange's technique, or by taking Laplace transforms, to yield

$$Q(z;t) = \exp\left(\frac{\alpha}{\mu} \sum_{i=1}^{\infty} q_i \sum_{j=1}^{i} \left[\frac{z^j}{j} - \frac{(1+(z-1)\mathrm{e}^{-\mu t})^j}{j}\right]\right). \tag{2.4}$$

Not only may moments and probabilities be derived directly from this solution, but Matthews *et al.* (2003) show that by making a judicious choice for the (factorial) mass immigration PGF, it is possible to tailor the process to take specific distributional forms.

A less opaque representation of result (2.4) can be developed by defining

$$\zeta_i(j) = \frac{1}{i} - \left(\frac{\mathrm{e}^{-\mu t}}{1-\mathrm{e}^{-\mu t}}\right)^i \sum_{k=1}^{j} \binom{k}{i} \frac{(1-\mathrm{e}^{-\mu t})^k}{k} \quad (j > 0) \tag{2.5}$$

with

$$\zeta_0(j) = -\sum_{i=1}^{j} \zeta_i(j). \tag{2.6}$$

For we can then write the generating function (2.4) in the form

$$Q(z;t) = \exp\left(\sum_{i=0}^{\infty} \xi_i z^i\right) \tag{2.7}$$

where

$$\xi_i = \frac{\alpha}{\mu}\sum_{j=i}^{\infty} q_j \zeta_i(j) \quad \text{for} \quad i > 0 \quad \text{and} \quad \xi_0 = -\sum_{j=1}^{\infty} \xi_j. \tag{2.8}$$

On denoting

$$Q_i(\lambda) \equiv \sum_{n=0}^{\infty} \frac{\lambda^n \mathrm{e}^{-\lambda}}{n!} z^{in} = \exp\{\lambda(z^i - 1)\}, \tag{2.9}$$

it then follows that $Q(z;t)$ has the pure product representation

$$Q(z;t) = Q_1(\xi_1)Q_2(\xi_2)\dots Q_i(\xi_i)\dots. \tag{2.10}$$

Taking logarithms and placing $z = e^{\theta}$ in the PGF (2.10) yields the associated cumulant generating function (CGF)

$$K(\theta;t) \equiv \sum_{i=1}^{\infty} \frac{\kappa_i \theta^i}{i!} = \sum_{i=1}^{\infty} \frac{\theta^i}{i!} \sum_{k=1}^{\infty} k^i \xi_k. \tag{2.11}$$

This is a particularly simple and transparent form, and it immediately follows that the ith cumulant

$$\kappa_i(t) = \xi_1 + 2^i \xi_2 + \dots + k^i \xi_k + \dots. \tag{2.12}$$

In particular, the mean

$$\kappa_1(t) = \frac{\alpha}{\mu}(1 - \mathrm{e}^{-\mu t}) \sum_{i=1}^{\infty} i q_i, \tag{2.13}$$

and the variance

$$\kappa_2(t) = \frac{\alpha}{2\mu} \sum_{i=1}^{\infty} i q_i [i(1 - \mathrm{e}^{-2\mu t}) + (1 - \mathrm{e}^{-\mu t})^2]. \tag{2.14}$$

Moreover, letting $t \to \infty$ in (2.13) and (2.14) gives

$$\kappa_1(\infty) = \frac{\alpha}{\mu} \sum_{i=1}^{\infty} i q_i \quad \text{and} \quad \kappa_2(\infty) = \frac{\alpha}{\mu} \sum_{i=1}^{\infty} q_i i(i+1). \tag{2.15}$$

So for the equilibrium mean and variance to exist we require $\sum_{i=1}^{\infty} i q_i < \infty$ and $\sum_{i=1}^{\infty} i(i+1) q_i < \infty$; in general, the existence of the rth cumulant $\kappa_r(\infty)$ requires $\sum_{i=1}^{\infty} i^r q_i < \infty$. This implies that by making a judicious choice for the $\{q_i\}$ we can generate processes for which only the first r limiting cumulants exist. For example, letting q_i involve the Riemann zeta function through

$$q_i = 1\Bigg/\left[i^{r+2}\sum_{j=1}^{\infty}\frac{1}{j^{r+2}}\right] \tag{2.16}$$

results in the first r cumulants being finite and the remainder infinite. For a comprehensive discussion of results for mass immigration (and annihilation) processes relating to existence, uniqueness, PGF, resolvent and equilibrium issues see Chen & Renshaw (1990, 1993*a*,*b*, 1995, 1997, 2000, 2004) and Renshaw & Chen (1997).

The probability $p_n(t)$ is found by extracting the coefficient of z^n in the generating function (2.4), and this calculation is facilitated by noting that

$$\exp\left(\sum_{i=0}^{\infty}\xi_i z^i\right) = \mathrm{e}^{\xi_0}\times\sum_{i_1=0}^{\infty}\xi_1\frac{z^{i_1}}{i_1!}\times\sum_{i_2=0}^{\infty}\xi_2\frac{z^{2i_2}}{i_2!}\times\ldots\times\sum_{i_k=0}^{\infty}\xi_k\frac{z^{ki_k}}{i_k!}\times\ldots.$$

For we immediately see that $p_0=\mathrm{e}^{\xi_0}$, whilst for $n>0$

$$p_n(t) = \sum_{j_n=0}^{[m_n/n]}\frac{\xi_n^{j_n}}{j_n!}\sum_{j_{n-1}=0}^{[m_{n-1}/n-1]}\frac{\xi_{n-1}^{j_{n-1}}}{j_{n-1}!}\cdots\sum_{j_3=0}^{[m_3/3]}\frac{\xi_3^{j_3}}{j_3!}\sum_{j_2=0}^{[m_2/2]}\frac{\xi_2^{j_2}}{j_2!}\frac{\xi_1^{m_1}}{m_1!}e^{\xi_0}, \tag{2.17}$$

where $[x]$ denotes the integer part of x, $m_i=0$ for $i>n$, $m_n=n$ and $m_i=m_{i+1}-(i+1)j_{i+1}$ for $i=1, 2, \ldots, n-1$. To determine the associated factorial moments, we first write (2.7) in the form

$$Q(1+z';t)\equiv\exp\left(\sum_{i=0}^{\infty}\xi_i(1+z')^i\right) = \exp\left(\sum_{i=1}^{\infty}\varphi_i(z')^i\right), \tag{2.18}$$

where $\varphi_0=0$ and

$$\varphi_i = \sum_{j=0}^{\infty}\binom{j}{i}\xi_j \tag{2.19}$$

for $i>0$. By extracting the coefficient of $(z')^r/r!$ it then follows that the rth factorial moment for the general k-tuple immigration case is

$$M_r(t) = \sum_{j_k=0}^{[m_r/r]}\frac{\varphi_k^{j_r}}{j_r!}\sum_{j_{k-1}=0}^{[m_{r-1}/r-1]}\frac{\varphi_{r-1}^{j_{r-1}}}{j_{r-1}!}\cdots\sum_{j_3=0}^{[m_3/3]}\frac{\varphi_3^{j_3}}{j_3!}\sum_{j_2=0}^{[m_2/2]}\frac{\varphi_2^{j_2}}{j_2!}\frac{\varphi_1^{m_1}}{m_1!}. \tag{2.20}$$

3. The associated counting process

Having developed the general population process, we are now in a position to consider the associated counting statistics. Suppose we replace basic 'death' by a two-type 'exit' process, namely each individual either dies unobserved at constant rate μ or else its death is observed (and hence counted) at rate η. So now there is an overall death rate $\delta=\mu+\eta$. Then the forward equation corresponding to (2.1) takes the bivariate form

$$\frac{\mathrm{d}p_{nc}(t)}{\mathrm{d}t} = \mu(n+1)p_{n+1,c}(t) + \eta(n+1)p_{n+1,c-1}(t) + \alpha\sum_{i=1}^{\infty} q_i p_{n-i,c}(t) - (\delta n+\alpha)p_{nc}(t), \quad (3.1)$$

where $n,\ c=0,\ 1,\ 2,\ \ldots$ and $p_{nc}(t)=0$ for $n,\ c=-1,-2,\ \ldots$. On defining the joint probability generating function

$$Q(z,s;t) \equiv \sum_{n,c=0}^{\infty} z^n s^c p_{nc}(t), \quad (3.2)$$

we obtain

$$\frac{\partial Q}{\partial t} + (\delta z - \eta s - \mu)\frac{\partial Q}{\partial z} = \alpha Q \sum_{i=1}^{\infty} q_i (z^i - 1). \quad (3.3)$$

On retaining the assumption that the population is of size zero at time $t=0$, this solves to yield

$$Q(z,s;t) = \exp\left(\sum_{i=1}^{\infty}\frac{\alpha q_i}{\delta^{i+1}}\sum_{r=1}^{i}\frac{1}{r}\binom{i}{i-r}(\eta s+\mu)^{i-r}(\delta z-\eta s-\mu)^r(1-e^{-r\delta t})\right) \times \exp\left(\sum_{i=1}^{\infty}\frac{\alpha t(\eta s+\mu)^i q_i}{\delta^i} - \alpha t\right). \quad (3.4)$$

Whence setting $z=1$ in (3.4) produces the marginal counting generating function

$$Q(1,s;t) = \exp\left(\sum_{i=1}^{\infty}\frac{\alpha q_i}{\delta^{i+1}}\sum_{r=1}^{i}\frac{1}{r}\binom{i}{i-r}(\eta s+\mu)^{i-r}\eta^r(1-s)^r(1-\mathrm{e}^{-r\delta t})\right) \times \exp\left(\sum_{i=1}^{\infty}\frac{\alpha t(\eta s+\mu)^i q_i}{\delta^i} - \alpha t\right). \quad (3.5)$$

This can be expressed in the more direct form

$$Q(1,s;t) = \exp\left(\sum_{i=0}^{\infty}\omega_i s^i\right) \quad (3.6)$$

where

$$\omega_i = \sum_{j=i}^{\infty} q_j \psi_i(j) \ \ (i>0) \ \ \text{with } \omega_0 = -\sum_{j=1}^{\infty}\omega_j, \quad (3.7)$$

and for $1 \le i \le j$

$$\psi_i(j) = \frac{\alpha}{\delta^{j+1}} \sum_{r=1}^{j} \frac{\eta^r (1 - e^{-r\delta t})}{r} \binom{j}{j-r} \sum_{n=0}^{i} \binom{r}{n} \binom{j-r}{i-n} \times (-1)^n \eta^{i-n} \mu^{j+n-i-r} + \binom{j}{i} \frac{\alpha \eta^i \mu^{j-i} t}{\delta^j} \tag{3.8}$$

with

$$\psi_0(j) = -\sum_{i=1}^{j} \psi_i(j). \tag{3.9}$$

The marginal counting probabilities $p_c(t)$ can now be found by extracting the coefficient of s^c in (3.6); whilst to obtain the marginal counting cumulants we successively differentiate (3.6) in the form $K(\cdot,\, \phi;\, t) = \ln[Q(\cdot,\, \mathrm{e}^{\phi};\, t)]$ and place $\phi = 0$. In particular, this yields the counting mean

$$\kappa_1^c(t) = \frac{\alpha \eta}{\delta^2} (\delta t + \mathrm{e}^{-\delta t} - 1) \sum_{i=1}^{\infty} i q_i, \tag{3.10}$$

and the counting variance

$$\kappa_2^c(t) = \frac{\alpha \eta}{\delta^3} \sum_{i=1}^{\infty} q_i i [\delta^2 t - (1 - \mathrm{e}^{-\delta t})(\eta i + \mu) + \eta (i-1)(\delta t - (1 - \mathrm{e}^{-\delta t})^2/2)]. \tag{3.11}$$

Moreover, the above general results subsume special cases developed previously, for example

basic single-immigration-death process	$q_1 = 1$ and $q_i = 0$ for $i \neq 1$
paired immigrants (Jakeman *et al.* 1995)	$q_2 = 1$ and $q_i = 0$ for $i \neq 2$
geometric immigration (Jakeman *et al.* 2002)	$q_i = (1-\xi)\xi^{i-1}$ for $0 < \xi < 1$
power-law process (Hopcraft *et al.* 2002)	$q_i = \nu \Gamma(i-\nu)/[\Gamma(1-\nu) i!]$ for $0 < \nu < 1$.

As two further cases are especially interesting, we shall now develop their associated results separately.

4. Special case 1: *k*-batch immigration

Let us first consider the case where each batch arrival comprises exactly k immigrants, so that $q_k = 1$ and $q_j = 0$ for $j \neq k$. Then the corresponding Kolmogorov forward equation

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \mu(n+1) p_{n+1}(t) + \alpha p_{n-k}(t) - (\mu n + \alpha) p_n(t) \tag{4.1}$$

solves to yield the PGF solution

$$Q(z;t) = \exp\left(\frac{\alpha}{\mu}\sum_{i=1}^{k}\frac{z^i - (1+(z-1)e^{-\mu t})^i}{i}\right). \tag{4.2}$$

We can immediately obtain the corresponding population size probabilities and moments as a special case of the general solutions (2.17) and (2.20).

As noted earlier, Jakeman *et al.* (1995) demonstrate that pure pair-immigration causes an odd–even (i.e. a 'saw-tooth up-down') effect in the probability and moment structures, while Gillespie & Renshaw (submitted) use a Hermite polynomial representation to show that pure-triple immigration generates triple saw-tooth behaviour in that probabilities $p_{3n}(t)$, $p_{3n+1}(t)$ and $p_{3n+2}(t)$ follow different 'trajectories' as n increases. To see that this extends naturally to the pure k-batch immigration-death process, we first note that since $q_k=1$ and $q_j=0$ for $j\neq k$, expression (2.8) reduces to $\xi_i=(\alpha/\mu)\zeta_i(k)$ for $i>0$. Whence inspection of (2.5) shows that $\xi_i>0$ for $1\leq i\leq k$ and $\xi_i=0$ otherwise. So expression (2.17) contains at most $k-1$ summation terms, with the last (i.e. leftmost) one being $\sum_{j_k=0}^{[m_k/k]}\xi_k^{j_k}/j_k!$, and it is this feature that causes the probabilities $\{p_{kn+j}(t)\}$ to generate different trajectories for $j=0, 1, \ldots, k-1$ as n increases through 0, 1, 2, …. Consider the basic $k=2$ case. Then solution (2.17) reduces to

$$p_n(t) = \sum_{j_2=0}^{[m_2/2]}\frac{\xi_2^{j_2}}{j_2!}\frac{\xi_1^{m_1}}{m_1!}e^{\xi_0}.$$

Hence, as the only terms in (2.17) that generate a non-zero contribution are those with $j_3=\ldots=j_n=0$, and as $m_i=m_{i+1}-(i+1)j_{i+1}$, it follows that $m_2=m_3=\ldots=m_n=n$. Moreover, as $m_1=m_2-2j_2=n-2j_2$, we have, on replacing n by $2n$ and $2n+1$, respectively, that

$$p_{2n}(t) = \sum_{j_2=0}^{n}\frac{\xi_2^{j_2}}{j_2!}\frac{\xi_1^{2n-2j_2}}{(2n-2j_2)!}e^{\xi_0} \tag{4.3}$$

and

$$p_{2n+1}(t) = \sum_{j_2=0}^{n}\frac{\xi_2^{j_2}}{j_2!}\frac{\xi_1^{2n-2j_2}}{(2n-2j_2)!}e^{\xi_0}\times\left(\frac{\xi_1}{2n+1-2j_2}\right), \tag{4.4}$$

from which the structure of the odd–even effect is immediately apparent. Similarly, on taking $k=3$ and $n=3s+r$ over $s=0, 1, 2, \ldots$ and $r=0, 1, 2$, we have

$$p_{3s+r}(t) = \sum_{j_3=0}^{s}\frac{\xi_3^{j_3}}{j_3!}\sum_{j_2=0}^{[(3(s-j_3)+r)/2]}\frac{\xi_2^{j_2}}{j_2!}\frac{\xi_1^{m_1}}{m_1!}e^{\xi_0} \tag{4.5}$$

where $m_1=3(s-j_3)+r-2j_2$. Thus, for a given s the dominant first summation remains unchanged over $r=0, 1, \ldots, k-1$; only the second summation is affected. Likewise, for general k, by taking $n=ks+r$ for $r=0, 1, \ldots, k-1$ we see that for given s the dominant summation

$$\sum_{j_k=0}^{s} \frac{\xi_k^{j_k}}{j_k!}$$

remains unchanged as r sweeps through 0, 1, ..., $k-1$, thereby giving rise to k-tuple saw-tooth behaviour centred around this driving term.

Not only does this probability structure induce a parallel effect on the corresponding moments, but k also makes an impact on the relative cumulant values. For it follows from (2.12) that the ith equilibrium cumulant is

$$\kappa_i(\infty) = \frac{\alpha}{\mu} \sum_{m=1}^{k} m^{i-1}, \tag{4.6}$$

whence the ratio

$$\frac{\kappa_i(\infty)}{\kappa_{i-1}(\infty)} = \frac{1 + 2^i + \ldots + k^i}{1 + 2^{i-1} + \ldots + k^{i-1}} \to k$$

as $i \to \infty$.

Further k-tuple properties can be generated by letting $t \to \infty$ in equation (4.1) to form the equilibrium equations for $\pi_n = p_n(\infty)$, namely

$$(\mu n + \alpha)\pi_n - \alpha\pi_{n-k} = \mu(n+1)\pi_{n+1}. \tag{4.7}$$

For on noting that $\pi_i = 0$ for $i < 0$ and $\pi_0 = Q(0;\ t)$, solving equation (4.7) recursively for $n = 0, 1, \ldots, k-1$ immediately yields

$$\pi_n = \frac{\Gamma(n + \alpha/\mu)}{\Gamma(n+1)\Gamma(\alpha/\mu)} \exp\left(-\frac{\alpha[\Psi(k+1) + \gamma]}{\mu}\right). \tag{4.8}$$

Here (e.g. Abramowitz & Stegun 1970)

$$\gamma = \lim_{m\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} - \ln(m)\right) = 0.57721\ldots$$

is Euler's constant,

$$\Psi(1) = \gamma \quad \text{and} \quad \Psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \quad \text{for } n \geq 2$$

denotes the Digamma function, and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

is the Gamma function. Whence after some algebra, substituting the specific solution (4.8) into the general equation (4.7) for $n = 1, 2, \ldots$ and $1 \leq m < k$,

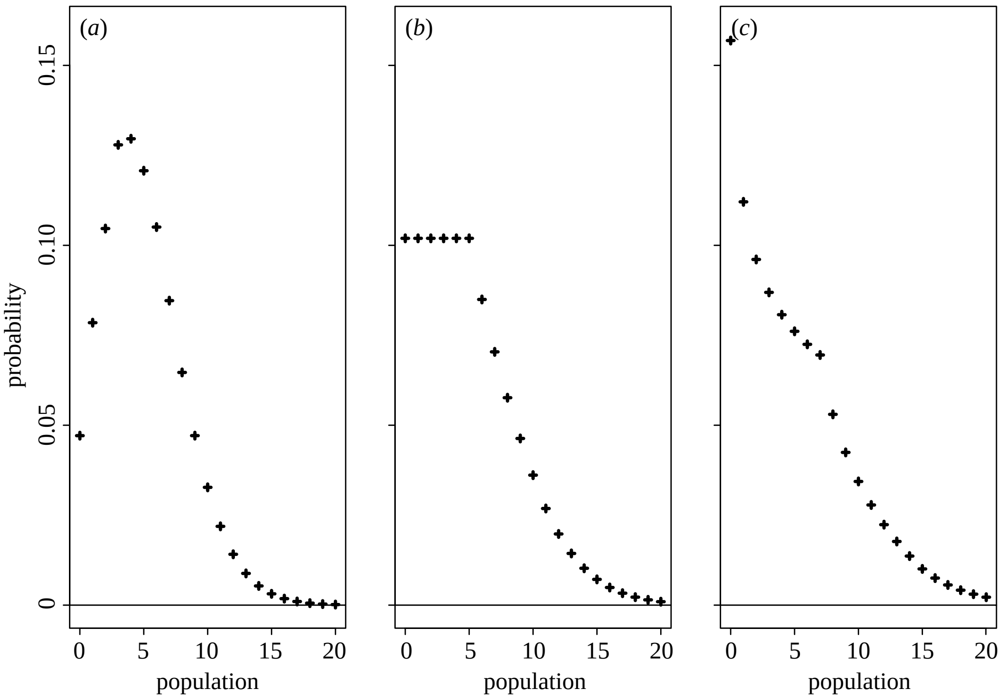

Figure 1. Plot of the theoretical equilibrium PDF $\{\pi_n\}$ corresponding to (4.9) for (*a*) $k=3$, (*b*) $k=5$ and (*c*) $k=7$ immigrants, with $\mu=1$ and $\alpha=5/k$. As the number of immigrants per batch increases, the distribution changes from a unimodal distribution with mode 5 to a PDF which is essentially J-shaped.

and reorganising the resulting expression, leads to the general solution

$$\pi_{nk+m} = \pi_{nk} \prod_{i=1}^{m} \frac{\mu(kn+i-1)+\alpha}{\mu(nk+i)} - \sum_{i=0}^{m-1} \frac{\alpha\pi_{(n-1)k+i}}{\mu(nk+m)} \times \prod_{j=i}^{m-2} \frac{\mu(nk+j+1)+\alpha}{\mu(nk+j+1)}, \tag{4.9}$$

which clearly exposes the underlying k-tuple equilibrium probability structure.

Although the equilibrium probabilities $\{\pi_{nk+m}\}$, never mind the time dependent probabilities $\{p_{nk+m}(t)\}$, clearly possess a highly opaque structure, they nevertheless expose some interesting characteristics. For example, if $\alpha=\mu$ then

$$\pi_n = \exp(-\Psi(k+1)-\gamma) \quad (n=0,1,\ldots,k), \tag{4.10}$$

that is, $\pi_0=\pi_1=\ldots=\pi_k$. This feature is illustrated in figure 1, where the equilibrium PDF is shown for batch size $k=3$, 5 and 7, individual death rate $\mu=1$, and the mean number of arrivals per unit time is fixed at 5, that is $\alpha=5/k$ for batch size k. So when $k=5$ (figure 1*b*), then $\alpha=\mu=1$ and $\pi_0=\pi_1=\ldots=\pi_5\simeq 0.102$, and this feature clearly delineates two quite different forms of the PDF.

When $k=3$ then $\alpha=5/k>\mu=1$ and $\{\pi_n\}$ follows a unimodal distribution with mode 5 (figure 1*a*); whilst for $k=7$ (figure 1*c*) the PDF is essentially J-shaped with a point of inflexion at k (yet another manifestation of the k-effect). Moreover, the figures indicate that as k increases, $p_0(\infty)\to 1$. Indeed, if $\alpha k/\mu=1$ then the equilibrium probability of an empty population is

$$\pi_0 = \exp(-[\Psi(k+1)+\gamma]/k)\to 1 \tag{4.11}$$

as $k\to\infty$. For as k increases, the waiting time between subsequent immigrants also increases, and as each individual member dies at the same rate this implies that a zero population size is present for longer time periods. On examining the associated cumulants, i.e. expression (4.6), we observe that $\kappa_1=1$, but for $i\geq 2$, $\kappa_i\to\infty$ as $k\to\infty$. This process therefore generates a series of population spikes, with immigrants entering the population in large quantities before quickly dying out, thereby leaving an empty population in between times.

Parallel results for the counting measures can be developed in a similar manner. For example, on deriving the marginal counting generating function by setting $q_k=1$ in Expression (3.5), we obtain

$$Q(1,s;t) = \exp\left(\frac{\alpha}{\delta^{k+1}}\sum_{r=1}^{k}\frac{1}{r}\binom{k}{k-r}(\eta s+\mu)^{k-r}\eta^r(1-s)^r(1-e^{-r\delta t})\right)$$
$$\times\exp\left(\frac{\alpha t(\eta s+\mu)^k}{\delta^k}-\alpha t\right). \tag{4.12}$$

Although extracting the associated counting probabilities is clearly a non-trivial pursuit, for large t the cumulants take the neat form

$$\kappa_i^c(t)\simeq\frac{\alpha\mu^k t}{\delta^k}\sum_{m=1}^{k}\binom{k}{m}\left(\frac{\eta}{\mu}\right)^m m^i. \tag{4.13}$$

Hence,

$$\frac{\kappa_{i+1}^c(t)}{\kappa_i^c(t)}\to\frac{k^{i+1}}{k^i}=k \tag{4.14}$$

as $i\to\infty$, which is identical to the limiting ratio of the population cumulants. Moreover, the rate at which (4.14) reaches its limit is directly dependent on the unobserved death rate μ. Note that when $\mu=0$ then

$$\kappa_i^c(t)\simeq\alpha t k^i, \tag{4.15}$$

whence it follows that $\kappa_{i+1}^c(t)/\kappa_i^c(t)\simeq k$ for all $i=1, 2, \ldots$.

5. Special case 2: 'Schoenberg'-batch immigration

In the previous sections we have demonstrated that k-tuple behaviour can exhibit similar manifestations in both the associated probability and moment structures. However, since an infinite number of probability measures can give rise to the same set of moments, there is no reason to presuppose that such similarity will hold universally. For our second example, we shall therefore exploit a classic example of this phenomenon due to Schoenberg (1983); for a comprehensive review of such probability classes see Stoyanov (1988).

Suppose we change the range of support for the Poisson distribution from the non-negative integers $n=0, 1, 2, 3, 4, \ldots$ to the non-exponentially bounded set $n=0, 1, 2, 4, 8, \ldots$, so that

$$p_n(t) = \begin{cases} e^{-2}2^m/m! & : \text{for } n = 2^m \quad (m = 0, 1, 2, \ldots), \\ 0 & : \text{otherwise.} \end{cases} \tag{5.1}$$

Then the corresponding rth raw moment is given by

$$\mu'_r = \sum_{m=0}^{\infty} e^{-2}2^{rm}2^m/m! = \exp[2(2^r - 1)]. \tag{5.2}$$

Now consider the function

$$h(z) \equiv \prod_{k=1}^{\infty}(1 - z/2^k), \tag{5.3}$$

and let it have a Taylor expansion around zero of

$$h(z) = \sum_{m=0}^{\infty} c_m z^m. \tag{5.4}$$

Then on noting the equality $h(2z)=(1-z)h(z)$, we have the relation $c_m/c_{m-1}=(1-2m)^{-1}$. So as $c_0=1$, it immediately follows that

$$c_m = (-1)^m[(2-1)(2^2-1)\ldots(2^m-1)]^{-1}. \tag{5.5}$$

Writing $a_m=m!c_m$, as $(2^m-1)\geq m$ we see that $|a_m|\leq 1$ for all m. Moreover, for $r=0, 1, \ldots$

$$e^{-2}\sum_{m=0}^{\infty} 2^{rm}a_m 2^m/m! = e^{-2}\sum_{m=0}^{\infty} c_m(2^m)^{r+1} = e^{-2}h(2^{r+1}) = 0. \tag{5.6}$$

Using the probabilities (5.1) as a base, suppose we now modify them by defining

$$\begin{aligned} p^{\varepsilon}_{2^m} &= p_{2^m}(1+\varepsilon a_m) \\ &= \frac{e^{-2}2^m}{m!}\{1+\varepsilon m!(-1)^m[(2-1)(2^2-1)\ldots(2^m-1)]^{-1}\} \end{aligned} \tag{5.7}$$

over $m=0, 1, 2, \ldots$ for $|\varepsilon|\leq 1$. Since we have already shown through expression (5.6) that

$$\sum_{m=0}^{\infty} \varepsilon a_m p_{2^m} = 0,$$

it immediately follows that

$$\sum_{m=0}^{\infty} p_{2^m}^{\varepsilon} = \sum_{m=0}^{\infty} p_{2^m}(1+\varepsilon a_m) = 1.$$

Moreover, as $|\varepsilon|\leq 1$ and $|a_m|\leq 1$, we have $p_{2^m}\geq 0$. So $\{p_{2^m}^{\varepsilon}\}$ is a proper probability distribution. Using (5.2) and (5.6) the associated raw moments are

$$\mu_r'^{\varepsilon} = \sum_{m=0}^{\infty} \frac{e^{-2}2^{rm}2^m}{m!}(1+\varepsilon a_m) = \mu_r' + 0, \tag{5.8}$$

and so $\{p_{2^m}^{\varepsilon}\}$ is a set of distributions with identical moments $\{\mu_r'\}$ but different probability structures.

Let us now inject this form into our stochastic model by retaining the death rate μ but allowing only batches of size 2^k immigrants to enter the population. Thus the immigration rate $q_i = (e^{-2}2^k/k!)(1+\varepsilon a_k)$ for $i=2^k$ over $k=0, 1, 2, \ldots$, and $q_i=0$ otherwise. Then the forward Kolmogorov equation (2.1) takes the form

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \mu(n+1)p_{n+1}(t) + \alpha\sum_{k=0}^{\infty}\frac{e^{-2}2^k}{k!}(1+\varepsilon a_k)p_{n-2^k}(t) - (\mu n+\alpha)p_n(t), \tag{5.9}$$

which yields the generation function solution

$$\begin{aligned} Q(z;t) &= \exp\left(\frac{\alpha}{\mu}\sum_{k=0}^{\infty}\frac{e^{-2}2^k}{k!}(1+\varepsilon a_k)\sum_{i=0}^{2^k}\zeta_i(2^k)z^i\right) \\ &= \exp\left(\frac{\alpha}{\mu}\sum_{k=0}^{\infty}\frac{e^{-2}2^k}{k!}(1+\varepsilon a_k)\sum_{i=1}^{2^k}\frac{z^i-(1+(z-1)e^{-\mu t})^i}{i}\right), \end{aligned} \tag{5.10}$$

on recalling that $\zeta_i(2^k)$ is given by (2.5) and $\zeta_0(2^k)$ by (2.6). Whence expanding (5.10) in powers of z^n yields the probabilities $\{p_n(t)\}$ as a special case of the general result (2.17).

To show that the associated moments are independent of ε, we parallel the general procedure (2.18)–(2.19) by substituting $z=1+z'$ into (5.10) to obtain

$$Q(1+z';t) = \exp\left(\frac{\alpha}{\mu}\sum_{k=0}^{\infty}\frac{e^{-2}2^k}{k!}(1+\varepsilon a_k)\sum_{i=1}^{2^k}\frac{1}{i}\sum_{j=0}^{i}\binom{i}{j}(1-e^{-j\mu t})(z')^j\right). \tag{5.11}$$

Whence it follows that

$$Q(1+z';t) = \exp\left(\sum_{m=1}^{\infty}\varphi_m(z')^m\right) \tag{5.12}$$

where

$$\varphi_m = \frac{\alpha}{\mu}\sum_{k=0}^{\infty}\frac{e^{-2}2^k}{k!}(1+\varepsilon a_k)\sum_{i=1}^{2^k}\frac{1}{i}\binom{i}{m}(1-e^{-m\mu t})$$
$$= \frac{\alpha(1-e^{-m\mu t})}{m\mu}\sum_{k=0}^{\infty}\frac{e^{-2}2^k}{k!}(1+\varepsilon a_k)\binom{2^k}{m}. \tag{5.13}$$

Then on noting the relationship (5.6) we obtain

$$\varphi_m = \frac{\alpha(1-e^{-m\mu t})}{m\mu}\sum_{k=0}^{\infty}\frac{e^{-2}2^k}{k!}\binom{2^k}{m}, \tag{5.14}$$

thereby proving that the factorial moments of the process (5.9) are independent of ε. Thus our system of Schoenberg–Poisson immigration-death processes generates an infinity of different probability distributions all of which possess the same moment structure. In particular, they share a common mean and variance, namely

$$\kappa_1(t) = \frac{\alpha e^2}{\mu}(1-\mathrm{e}^{-\mu t}) \quad \text{and} \quad \kappa_2(t) = \frac{\alpha e^2}{2\mu}[(1-\mathrm{e}^{-\mu t})^2 + e^4(1-\mathrm{e}^{-2\mu t})]. \tag{5.15}$$

Figure 2 depicts the three equilibrium probability structures $\{\pi_n^{\varepsilon}\}$ for the two extreme values $\varepsilon=-1$ and 1, together with the centre value $\varepsilon=0$, and shows how π_n^{-1} and π_n^1 both oscillate around, and bound, π_n^0. While a simulation (figure 3) of the process for $\varepsilon=0$ highlights the strong population surges induced by a sudden influx of fresh immigrants (here of size 64 at times $t\simeq 10$ and 65) which are immediately followed by their swift decline. It is the existence of these surges that causes the variance, $e^2(1+e^4)/2$, to be large in comparison to the overall mean, e^2.

Now the bivariate Kolmogorov forward equation (3.1), namely

$$\frac{\mathrm{d}p_{nc}(t)}{\mathrm{d}t} = \eta(n+1)p_{n+1,c-1}(t) + \mu(n+1)p_{n+1,c}(t)$$
$$+ \alpha\sum_{k=0}^{\infty}\frac{e^{-2}2^k}{k!}(1+\varepsilon a_k)p_{n-2^k,c}(t) - [\delta n+\alpha]p_{nc}(t), \tag{5.16}$$

where $\delta=\mu+\eta$, yields the solution

$$Q(z,s;t) = \exp\left(\alpha t e^{-2}\sum_{k=0}^{\infty}\frac{2^k}{k!}\frac{(\eta s+\mu)^{2^k}}{\delta^{2^k}}(1+\varepsilon a_k) - \alpha t\right)$$
$$\times\exp\left(\sum_{k=0}^{\infty}\frac{\alpha e^{-2}2^k}{k!\delta^{2^k+1}}\sum_{r=1}^{2^k}\frac{1}{r}\binom{2^k}{2^k-r}(\eta s+\mu)^{2^k-r}(\delta z-\eta s-\mu)^r(1-e^{-r\delta t})\right) \tag{5.17}$$

for the associated PGF (3.4). Whence on placing $z=1$ we obtain the marginal

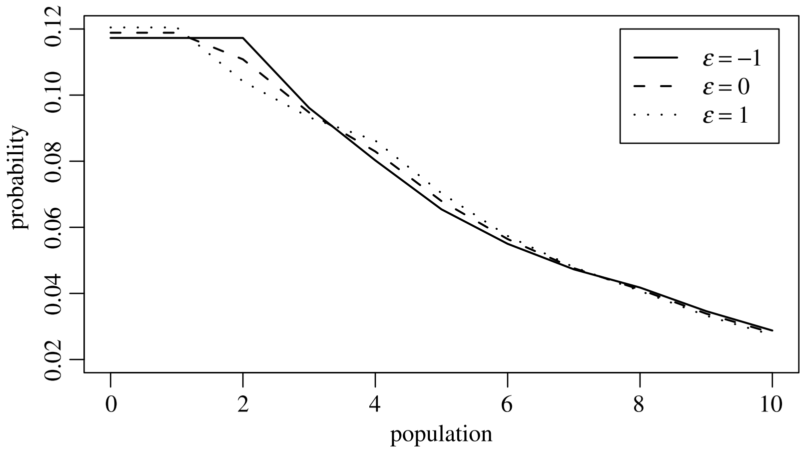

Figure 2. The equilibrium probability distribution of the Schoenberg–Poisson immigration-death process corresponding to (5.9), where $\alpha=1$, $\mu=1$ and $\varepsilon=-1$, 0, 1. The moments corresponding to each distribution are identical, that is they are independent of ε.

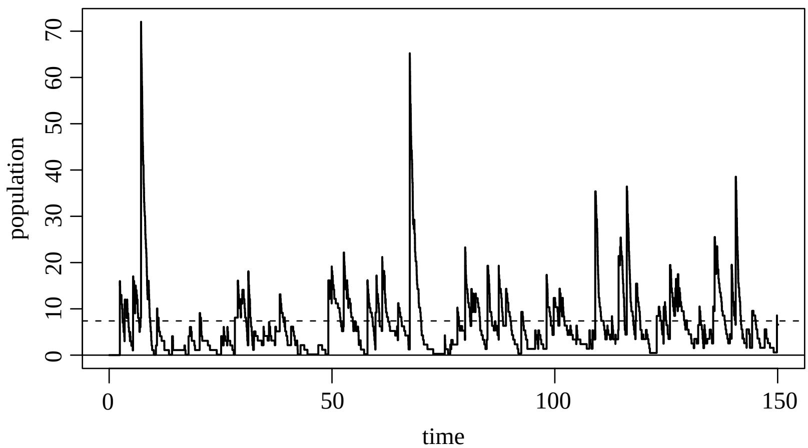

Figure 3. Stochastic simulation of the Schoenberg-Poisson immigration-death process corresponding to (5.9) for $n_0=0$, $\alpha=1$, $\mu=1$ and $\varepsilon=0$, and the superimposed equilibrium value (dashed line). The equilibrium distribution for this simulation is shown in figure 2.

counting PGF

$$Q(1,s;t)=\exp\left(\sum_{k=0}^{\infty}\frac{e^{-2}2^k}{k!}(1+\varepsilon a_k)\sum_{i=0}^{2^k}\psi_i(2^k)s^i\right). \qquad (5.18)$$

The marginal counting probability distribution can now be constructed from (5.18) using the general result (3.5). Moreover, on substituting $s=1+s'$ into the generating function (5.18) and using similar methods to those in § 4, we discover

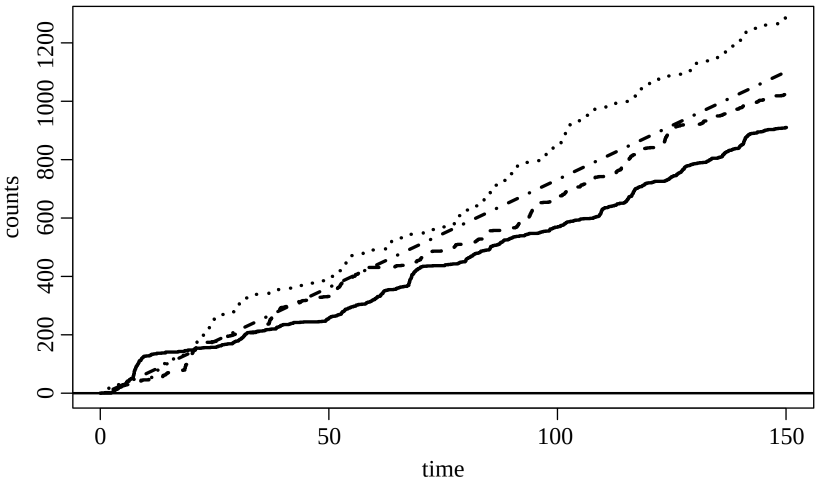

Figure 4. Three stochastic simulations of the counting process for $n_0=0$, $\alpha=1$, $\mu=0$, $\eta=1$ and $\varepsilon=0$, and the associated expected value (dashed dotted line); the simulation (solid line) corresponds to the counts obtained from figure 3. The moments for the counting process are also independent of ε.

that moments are once again identical for all values of ε. In particular, the mean

$$\kappa_1(t) = \frac{\alpha\eta e^2}{\delta^2}(\delta t + e^{-\delta t} - 1), \tag{5.19}$$

whilst the variance

$$\kappa_2(t) = \frac{\alpha\eta e^2}{\delta^3}[\delta^2 t - (1-e^{-\delta t})(\eta e^4 + \mu) + \eta(e^4 - 1)(\delta t - (1-\mathrm{e}^{-\delta t})^2/2)]. \tag{5.20}$$

Figure 4 shows three simulations for the counting process, one of which corresponds to the population size simulation shown in figure 3, together with $\kappa_1(t)$. Note how the two large population spikes corresponding to the batch immigration of 64 individuals at times $t \simeq 10$ and 65 carry over to the counting process.

6. Summary and conclusions

Although complex stochastic systems often have the potential for exhibiting extremely rich dynamic behaviour, gaining a direct understanding of such behavioural structure may not be possible if the system itself remains hidden. However, suppose we can record emission counts, that is the exit times of 'escaping individuals', whose underlying rate is dependent on population size. Then it is reasonable to presume that fluctuations in the internal (hidden) population will be at least partially transferred to this external series of countable events. Gillespie & Renshaw (submitted) pose the fascinating question of whether it is possible to determine the fundamental structure of a hidden stochastic process purely from counts of escaping individuals. They provide a partial answer by considering a double and triple immigration-death process, which is of immense importance in quantum optics. This present paper builds on these results by developing the probability and moment structure of an immigration-death process whose immigrants arrive in batches of size i at rate αq_i (i=1, 2, …). The extent to which information on the hidden population structure can be inferred from knowledge of the count times turns out to be considerable, and two special cases are considered which highlight the degree to which this can occur. First, we suppose that each immigration event comprises exactly k immigrants, so that $q_k=1$. Then, the associated stochastic measures, namely population size and count probabilities, together with their moments, follow directly from the general case, and neatly demonstrate that the 'k-tuple' population size effect, in which the probabilities $p_{kn}(t)$, $p_{kn+1}(t)$, …, $p_{kn+k-1}(t)$ follow different trajectories as n increases, carries over directly to the counting measures. Second, having demonstrated that this k-tuple behaviour can exhibit similar manifestations in both the associated probability and moment structures, we question whether such similarity might hold universally. Choosing q_i to follow the Schoenberg–Poisson form on i=1, 2, 4, 8, …, for which an infinity of population size PDF's generate an identical moment structure, we discover that exactly the same phenomenon holds true for the counting process. Future research could usefully focus on correlation properties of these processes. In particular, for the extreme 'Schoenberg' example, statistical correlation measures could be constructed using some form for 'clipping', since the large fluctuations in the number of individuals may saturate any detector (Matthews *et al.* 2003). Overall, the close link between population and counting measures truly runs deep, which bodes well for inferring hidden population structures from external counts.

A natural, albeit relatively straightforward, way of exploiting this general relationship still further would be to include births, thereby extending the birth–death-counting results of Gillespie & Renshaw (submitted). However, a potentially far more profitable, and timely, direction for future research would be to place the mass annihilation-immigration process (see Chen & Renshaw 1990, 1993*a*,*b*, 1995, 1997, 2000, 2004; Renshaw & Chen 1997) within this counting process setting. For not only would this tap into a rich seam of probability theory, but by examining specific cases based around power-law distributions it would draw the respective fields of stochastic population theory and quantum optics much closer together.

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