ELSEVIER
Stochastic Processes and their Applications 88 (2000) 177–193

stochastic processes and their applications

www.elsevier.com/locate/spa

Existence, recurrence and equilibrium properties of Markov branching processes with instantaneous immigration

Anyue Chen[a,*], Eric Renshaw[b]

[a] *School of Computing and Mathematical Sciences, The University of Greenwich, Greenwich Maritime Campus, 30 Park Row, Greenwich, London SE10 9LS, UK*

[b] *Department of Statistics and Modelling Science, Livingstone Tower, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, UK*

Received 30 January 1998; received in revised form 21 January 1999; accepted 22 December 1999

Abstract

Attention has recently focussed on stochastic population processes that can undergo total annihilation followed by immigration into state j at rate α_j. The investigation of such models, called Markov branching processes with instantaneous immigration (MBPII), involves the study of existence and recurrence properties. However, results developed to date are generally opaque, and so the primary motivation of this paper is to construct conditions that are far easier to apply in practice. These turn out to be identical to the conditions for positive recurrence, which are very easy to check. We obtain, as a consequence, the surprising result that any MBPII that exists is ergodic, and so must possess an equilibrium distribution. These results are then extended to more general MBPII, and we show how to construct the associated equilibrium distributions.

Keywords: Annihilation; Catastrophe; Equilibrium distribution; Existence; Markov branching process; q-matrix; Recurrence; Resolvent; State-dependent immigration

1. Introduction

Markov branching processes (MBP) occupy a major niche in the theory and application of stochastic processes; standard references are Harris (1963), Athreya and Ney (1972) and Asmussen and Hering (1983), whilst for recent developments see Athreya and Jagers (1996). Within this framework both immigration and emigration have important roles to play, and the former can be traced to Foster (1971) and Pakes (1971) who consider a discrete branching process with immigration occurring only when the process occupies state zero. Yamazato (1975) investigated the continuous version in which once the process reaches zero it remains there for an exponentially distributed time and then jumps to state j with rate α_j; in this model the condition $\sum \alpha_j < \infty$ is imposed and is necessary for the related treatment.

* Corresponding author. Tel.: +44-20-8331-8492; fax: +44-20-8331-8665.
E-mail address: a.chen@greenwich.ac.uk (A. Chen).

0304-4149/00/$ - see front matter
PII: S0304-4149(99)00125-8

More recently, Chen and Renshaw (1990, 1993a, 1995) consider a new type of immigration process, namely instantaneous immigration. Specifically, the infinitesimal behaviour of the process is described by a q-matrix $Q=\{q_{ij};\ i,j\geqslant 0\}$ which splits into two parts Q_1 and Q_2. The first follows a standard branching process with $q^{(1)}_{i,i+r-1}=ib_r$ for $r=0,1,\dots$ (so b_0 corresponds to an individual death); whilst the second allows for total population destruction at rate $q^{(2)}_{i0}=ia$ $(i>0)$ followed by immigration into state j at rate $q^{(2)}_{0j}=\alpha_j$ $(j>0)$. Placing $\sum_{j=1}^{\infty}\alpha_j=\infty$ gives rise to the field of Markov branching processes with instantaneous immigration (MBPII). We refer to Pakes (1993) for an interesting discussion on the relationship between these two types of processes.

For Yamazato's process the corresponding q-matrix is stable and so belongs to the canonical case. In particular, there is no problem concerning existence since the Feller minimal process always exists. However, for MBPII, since the q-matrix is not stable, many standard techniques used in general Markov process theory are difficult to apply; few results have been obtained for the unstable case. Indeed, it is not known in general whether there exists a Markov process for an unstable pre-q-matrix (for details see Chen and Renshaw, 1993b). So the MBPII scenario provides a highly interesting and challenging problem in terms of developing general conditions under which an MPBII process will exist.

Results directly relevant to the above unstable pre-q-matrix Q include the following.

Proposition 1 (Chen and Renshaw, 1990, 1993a). *If $a=0$ then a MBPII exists if and only if*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\alpha_j\phi^*_{jk}(\lambda)<\infty \tag{1.1}$$

for some (and therefore for all) $\lambda>0$; $\phi^*(\lambda)=\{\phi^*_{ij}(\lambda);i,j\geqslant 0\}$ is the minimal resolvent associated with the stable q-matrix Q_1 (i.e. the underlying Markov process without immigration). If (1.1) holds then there are uncountably many MBPII of which only one is honest. Moreover, this honest process is always recurrent, and is positive recurrent if and only if

$$\int_0^1 \frac{h(q)-h(s)}{u(s)}\,\mathrm{d}s<\infty, \tag{1.2}$$

where

$$h(s)=\sum_{j=1}^{\infty}\alpha_j s^j\quad(|s|<1),\qquad u(s)=\sum_{j=0}^{\infty}b_j s^j\quad(|s|\leqslant 1) \tag{1.3}$$

and q is the extinction probability of the underlying MBP without immigration associated with Q_1.

Proposition 2 (Chen and Renshaw, 1995). *If $a>0$ then a MBPII exists if and only if*

$$\sum_{n=1}^{\infty}(\alpha_n/n)<\infty. \tag{1.4}$$

Under this condition there are uncountably many MBPII, exactly one of which is honest; this honest process is always positive recurrent.

As $\{\alpha_j\}$ and $\{b_j\}$ are likely to have algebraically amenable forms, conditions (1.2) and (1.4) should be fairly easy to verify. However, condition (1.1) is far more opaque since it is based on the resolvent $\phi^*(\lambda)$, rather than Q itself, and this is relatively hard to work with. Pakes (1993), for example, provides examples of null- and positive-recurrent MBPII, and although he uses the checking condition (1.2) the existence condition (1.1) is taken for granted. Hence it is necessary to answer the following two questions. (1) Can we find an equivalent condition to (1.1) that is easier to check? (2) What gap (if any) exists between the existence condition (1.1) and the positive recurrence condition (1.2); can we derive examples of processes that satisfy (1.1) but not (1.2)? This last question is equivalent to asking whether we can find a null-recurrent MBPII.

Rather surprisingly, the existence condition (1.1) is simply the positive recurrence condition (1.2), whence it follows that the answers to the above two questions are yes and no, respectively. The main results are detailed in Section 2, with proofs being given in Section 3. The examples provided in Section 4 show how easy our results are to apply, whilst Section 5 develops existence and recurrence results for general MBPII. Since any general MBPII is positive recurrent and irreducible (see Theorem 8), the equilibrium distribution must exist, and a discussion of equilibrium results is presented in Section 6.

2. Results

Theorem 1. *For the Q in Section* 1, *if $a=0$ then the following statements are equivalent*:

(i) *a Q-function exists*;
(ii) *a MBPII exists*;
(iii) *condition* (1.1) *holds true*;
(iv) $h(s)<\infty$ *when* $|s|<1$ *and for some* (*and hence for all*) $\varepsilon\in(q,1)$

$$\int_\varepsilon^1 \frac{h(s)}{u(s)}\,\mathrm{d}s > -\infty, \tag{2.1}$$

where $q<1$ is the unique root of $u(s)=0$ on $[0,1)$;
(v) $h(s)<\infty$ *for* $|s|<1$ *and*

$$\int_0^1 \frac{h(q)-h(s)}{u(s)}\,\mathrm{d}s < \infty. \tag{2.2}$$

Moreover, *when any of the above five conditions holds true*, *there are uncountably many MBPII. Exactly one of these processes is honest*, *and this honest MBPII is positive recurrent.*

Remark 1. Although parts (i) and (ii) of Theorem 1 are equivalent, the constructs of Q-function and MBPII may differ. Indeed, although a MBPII with q-matrix Q must be a Q-function, the converse does not necessarily follow. Specifically, although the honest MBPII is unique, there might exist infinitely many honest Q-functions (see Chen and Renshaw, 1990, and also the following Remark 6 for details).

Remark 2. In parts (iv) and (v) of Theorem 1 we require $h(s)$ to be convergent (follows from $h(1) \equiv \sum_{j=1}^{\infty} \alpha_j = \infty$). Also, conditions (2.1) and (2.2) imply that

$$\int_{\varepsilon}^{1} \frac{\mathrm{d}s}{u(s)} > -\infty, \tag{2.3}$$

since $h(s) \to \infty$ as $s \to 1$, which is precisely the Harris condition for the dishonesty of the underlying Markov branching process without immigration (MBP). That is, it implies that this underlying process is explosive. Thus $u'(1) = \infty$ and $u(s) = 0$ must have a (unique) root q on $[0, 1)$, this being the associated extinction probability. This confirms the close link between our MBPII existence conditions and the Harris honesty condition for the simple branching process. Note that condition (2.3) is particularly useful, since if $u(s)$ does not satisfy it then we may immediately deduce that no MPBII exists, the form of $h(s)$ being irrelevant.

Remark 3. The integrand of (2.2) is non-negative, since $u(s)$ is positive on $[0, q)$ but negative on $(q, 1)$ and $h(s) \geqslant 0$ is increasing over $[0, 1)$, whilst the integrands of (2.1) and (2.3) are non-positive.

Remark 4. Conditions (2.1) and (2.2) involve only the generating functions of $\{\alpha_j\}$ and $\{b_j\}$, and so satisfy our requirement of being easy to verify. In particular, since (2.1) just involves the behaviour of the integrand near 1, we do not need to evaluate the integral in order to verify that it converges.

Remark 5. It is easy to see that (2.1) is equivalent to the criterion

$$\int_{\varepsilon}^{1} \frac{h(s)}{1 - f(s)}\, \mathrm{d}s < \infty, \tag{2.4}$$

where $f(s) = \sum_{j \neq 1} (-b_j/b_1) s^j$ denotes the offspring p.g.f. For on considering $f'(1) = \infty$, which is equivalent to $u'(1) = \infty$ (see Remark 2), we have

$$\lim_{s \to 1} \frac{-u(s)}{1 - f(s)} = -b_1 \lim_{s \to 1} \frac{s - f(s)}{1 - f(s)} = -b_1 \lim_{s \to 1} \left(1 - \frac{1}{f'(s)}\right) = -b_1,$$

and so (2.1) and (2.4) are equivalent.

Note that it immediately follows from (2.4) that

$$\infty > \int_{\varepsilon}^{1} \frac{h(s)}{1 - f(s)}\, \mathrm{d}s > (1 - f(\varepsilon))^{-1} \int_{\varepsilon}^{1} h(s)\, \mathrm{d}s,$$

whence

$$\sum_{n=1}^{\infty} \alpha_n / n < \infty$$

which recovers Corollary 5.4 in Chen and Renshaw (1990).

If we compare (2.2) in Theorem 1 and (1.2) in Proposition 1 we immediately obtain the answer to Question (2) of Section 1. Moreover, it is important to note that we may consider far more general MBPII processes than that invoked in Theorem 1. Let the infinitesimal q-matrix Q take the form

$$Q = Q_1 + Q_2, \tag{2.5}$$

where $Q_1 = \{q_{ij}^{(1)}\}$ is still the standard branching generator

$$q_{ij}^{(1)} = \begin{cases} ib_{j-i+1} & \text{if } j \geqslant i-1, \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

with $b_j \geqslant 0$ $(j \neq 1)$ and $0 < -b_1 = \sum_{j \neq 1} b_j < \infty$, but now $Q_2 = \{q_{ij}^{(2)}\}$ takes the completely general form

$$q_{ij}^{(2)} = \begin{cases} -\infty & \text{if } i = j = 0, \\ \alpha_j & \text{if } i = 0,\ j \geqslant 1, \\ a_i & \text{if } i \geqslant 1,\ j = 0, \\ -a_i & \text{if } i = j \geqslant 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

where $\alpha_j \geqslant 0$ $(j \geqslant 1)$, $\sum_{j=1}^{\infty} \alpha_j = \infty$ and $a_i \geqslant 0$. Note that when $a_i = ia$ we recover the special cases discussed in Propositions 1 and 2. In order to develop this general case we need the following definition.

Definition. A MBPII is a denumerable Markov process on the state space $E = \{0, 1, \ldots\}$ whose transition function $P(t) = \{p_{ij}(t);\ i, j \geqslant 0\}$ satisfies

$$\lim_{t \to 0^+} [(p_{00}(t) - 1)/t] = -\infty \tag{2.8}$$

and

$$\mathrm{d}p_{ij}(t)/\mathrm{d}t = \sum_{k=0}^{\infty} p_{ik}(t) q_{kj} \quad (i \geqslant 0,\ j \geqslant 1,\ t \geqslant 0), \tag{2.9}$$

where $Q = \{q_{ij}\}$ is given in (2.5)–(2.7).

It then follows from (2.8) and (2.9) that

$$\lim_{t \to 0} [(P(t) - I)/t] = Q, \tag{2.10}$$

so $P(t)$ is the transition function of some MBPII. Note that (2.9) only holds true for $j \geqslant 1$, since when $j = 0$ we have $q_{00} = -\infty$ and it is not possible to write down the Kolmogorov equation.

Remark 6. On comparing (2.8) and (2.9) with (2.10) we see that although the transition function of a MBPII is a Q-function, the converse does not always hold true (see Remark 1).

We can generalise Theorem 1 to prove that for any Q defined by (2.5)–(2.7), if there exists a MBPII then the honest MBPII must be unique and positive-recurrent (Section 5). Note that this theorem contains the corresponding statements in Theorem 1 as corollaries. The associated existence result is as follows. If $\{a_j;\ j \geqslant 1\}$ are bounded, then there exists a MBPII if and only if any of the conditions in Theorem 1 hold true.

3. Proof of Theorem 1

The proof of (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follows from Proposition 1; for details see Chen and Renshaw (1990). To prove (iii) $\Leftrightarrow$ (iv) we first note (also by Chen and Renshaw, 1990) that condition (iii) implies that $h(s) < \infty$ when $|s| < 1$. Let $I(\lambda)$ denote the left-hand side of (1.1), and $P^*(t) = \{p^*_{ij}(t)\}$ the transition function of the underlying MBP. Then by the branching property

$$\sum_{k=1}^{\infty} p^*_{jk}(t) = \left(\sum_{k=0}^{\infty} p^*_{1k}(t)\right)^j - (p^*_{10}(t))^j = (\sigma(t))^j - (q(t))^j,$$

where $\sigma(t) \hat{=} \sum_{k=0}^{\infty} p^*_{1k}(t) < 1$ ($\forall t > 0$), since we have already proved that the underlying MBP is explosive, and $q(t) \hat{=} p^*_{10}(t) < \sigma(t)$ ($\forall t \geqslant 0$). Thus

$$I(\lambda) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \alpha_j \int_0^{\infty} e^{-\lambda t} p^*_{jk}(t)\, dt = \int_0^{\infty} e^{-\lambda t} \sum_{j=1}^{\infty} \alpha_j \sum_{k=1}^{\infty} p^*_{jk}(t)\, dt$$

$$= \int_0^{\infty} e^{-\lambda t} \sum_{j=1}^{\infty} \alpha_j [(\sigma(t))^j - (q(t))^j]\, dt = \int_0^{\infty} e^{-\lambda t}[h(\sigma(t)) - h(q(t))]\, dt. \tag{3.1}$$

The last equality holds true since $h(s)$ is well-defined for all $|s| < 1$. Now for any $\delta > 0$ we can write

$$I(\lambda) = \int_0^{\delta} e^{-\lambda t}[h(\sigma(t)) - h(q(t))]\, dt + \int_{\delta}^{\infty} e^{-\lambda t}[h(\sigma(t)) - h(q(t))]\, dt$$
$$\equiv I_1(\lambda) + I_2(\lambda), \tag{3.2}$$

and it is well-known that $\sigma(t) \downarrow$ as $t \to \infty$ whence $h(\sigma(t)) \downarrow$ since the function $h(\cdot)$ itself is increasing. So as $h(q(t)) \geqslant 0$,

$$I_2(\lambda) \leqslant \int_{\delta}^{\infty} e^{-\lambda t} h(\sigma(t))\, dt \leqslant h(\sigma(\delta)) \int_{\delta}^{\infty} e^{-\lambda t}\, dt = h(\sigma(\delta)) e^{-\lambda\delta}/\lambda < \infty, \tag{3.3}$$

since $\sigma(\delta) < 1$ whence $h(\sigma(\delta)) < \infty$. Thus $I(\lambda) < \infty$ if and only if $I_1(\lambda) < \infty$. However, $I_1(\lambda) < \infty$ if and only if

$$\int_0^{\delta} h(\sigma(t))\, dt < \infty. \tag{3.4}$$

Indeed,

$$\int_0^{\delta} e^{-\lambda t} h(\sigma(t))\, dt = I_1(\lambda) + \int_0^{\delta} e^{-\lambda t} h(q(t))\, dt, \tag{3.5}$$

since $q(t) \uparrow$ as $t \uparrow$, and so the last term in (3.5) is finite; for it is less than the finite value $h(q(\delta))(1 - e^{-\lambda\delta})/\lambda$. Thus $I_1(\lambda) < \infty$ if and only if

$$\int_0^{\delta} e^{-\lambda t} h(\sigma(t))\, dt < \infty. \tag{3.6}$$

However, (3.4) and (3.6) are equivalent, since

$$e^{-\lambda\delta} \int_0^{\delta} h(\sigma(t))\, dt \leqslant \int_0^{\delta} e^{-\lambda t} h(\sigma(t))\, dt \leqslant \int_0^{\delta} h(\sigma(t))\, dt.$$

Whence $I_1(\lambda) < \infty$ if and only if (3.4) holds true. But $\sigma(t)$ satisfies the backward Kolmogorov equation for the MBP, i.e.

$$\mathrm{d}\sigma(t)/\mathrm{d}t = u(\sigma(t)),$$

whence setting $s = \sigma(t)$ in (3.4) gives $I(\lambda) < \infty$ if and only if

$$\int_1^{\varepsilon} \frac{h(s)}{u(s)}\,\mathrm{d}s < \infty,$$

where $\varepsilon = \sigma(\delta) \in (q, 1)$. This completes the proof of (iii) $\Leftrightarrow$ (iv).

We now claim that (iv) $\Leftrightarrow$ (v). First note by Remark 2 that (2.1) implies (2.3), which, on combining with $q < 1$ and hence $h(q) < \infty$, in turn implies that (2.1) is equivalent to

$$\int_{\varepsilon}^{1} \frac{h(q) - h(s)}{u(s)}\,\mathrm{d}s < \infty \tag{3.7}$$

for some (and hence for all) $\varepsilon \in (q, 1)$. Comparing (3.7) with (2.2), and noting that q is the unique root of $u(s)$ on $[0, 1)$, then shows that it is sufficient to prove that the non-negative function $W(s) \equiv [h(q) - h(s)]/u(s)$ of s on $(0, 1)$ remains bounded as $s \to q$. Now $0 \leqslant q < 1$ and $q = 0$ if and only if $b_0 = 0$. Suppose $q > 0$. Then since q is the single root of $u(s) = 0$, we must have $u(s) = (q - s)g(s)$, for some function $g(s)$, where $g(q) \neq 0$ (in fact $g(q) > 0$). Moreover, by the Mean Value theorem, $h(s) - h(q) = (s - q)h'(\xi)$ where $q < \xi < s$ when $q < s$, and $h(q) - h(s) = (q - s)h'(\xi)$, where $s < \xi < q$ when $q > s$. It is easy to see that both $h'(s)$ and $g(s)$ are continuous functions of $s \in (0, 1)$, whence

$$\lim_{s\to q}\left[\frac{h(q) - h(s)}{u(s)}\right] = \lim_{s\to q}\left|\frac{(q - s)h'(\xi)}{(q - s)g(s)}\right| = \lim_{s\to q}\left|\frac{h'(\xi)}{g(s)}\right| = \frac{h'(q)}{g(q)},$$

since when $s \to q$, $\xi \to q$. Thus as $h'(q)$ is finite and $g(q) \neq 0$ (both are positive), we see that $h'(q)/g(q)$ is finite. It then follows that $[h(q) - h(s)]/u(s)$ (which is always positive) is bounded when $s \to q$. A similar proof holds for $q = 0$, with $s \to q$ being replaced by $s \to 0^+$ and $h(q) = 0$.

Finally, we need to prove that the unique and honest MBPII is always positive recurrent. First note that (2.2) (part (v) of Theorem 1) is exactly the same as the positive recurrence condition (1.2) of Proposition 1. Since the proof of the latter assumes that the MBPII is irreducible, it therefore remains for us to prove that the MBPII with all the catastrophe rates $a_i \equiv 0$ is irreducible. If $b_0 > 0$, so that all death rates are positive, then the process must clearly be irreducible (we have excluded the trivial case of all birth rates $b_2, b_3, \ldots$ being zero). Whilst even if $b_0 = 0$, and thus $q = 0$, the honest MBPII is still irreducible. To prove this we only need to show that $r_{k0}(\lambda) > 0$ for all $k > 0$, where $\{r_{k0}(\lambda)\}$ are the resolvent elements of the honest MBPII. Now by the construction theorem of this honest MBPII (see Chen and Renshaw, 1990) we know that

$$r_{k0}(\lambda) = r_{00}(\lambda)\left(1 - \lambda\sum_{j=1}^{\infty}\phi_{kj}^{*}(\lambda)\right), \tag{3.8}$$

where $\Phi^*(\lambda)=\{\phi^*_{ij}(\lambda)\}$ is the minimum Feller Q_1-resolvent associated with the MBP without immigration. However, the existence condition for MBPII implies that Q_1 is not regular, and hence that $\Phi^*(\lambda)$ is not honest. So we therefore have

$$1-\lambda\sum_{j=1}^{\infty}\phi^*_{kj}(\lambda)>0 \quad (\forall k,\lambda>0). \tag{3.9}$$

But $r_{00}(\lambda)$ is positive for all $\lambda>0$, and so the $r_{k0}(\lambda)$ in (3.8) are also always positive. This completes the proof of Theorem 1. □

4. Examples

We shall now demonstrate the ease of applying Theorem 1 by considering four specific examples; remember that for a MBPII $\sum_n \alpha_n=\infty$ and $u'(1)=\infty$.

Example 1. For $(\beta)_r=\Gamma(\beta+r)/\Gamma(\beta)$ $(r=1,2,\ldots)$ with $(\beta)_0=1$, consider

$$\alpha_n=[(\beta)_n]/n! \quad \text{and} \quad b_n=[(-m-1)_n-(1-c)(-m)_n]/n!, \tag{4.1}$$

where $\beta>0$, $0<c<1$ and $0<m<1$. It is easy to see that $\sum_{n=0}^{\infty}b_n=0$, $\sum_{n=2}^{\infty}b_n=(1-c)+cm<\infty$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$. Using (4.1) we obtain

$$h(x)\equiv\sum_{n=1}^{\infty}\alpha_n x^n=(1-x)^{-\beta}-1 \quad \text{and} \quad u(x)\equiv\sum_{n=0}^{\infty}b_n x^n=(c-x)(1-x)^m.$$

Thus $h(x)<\infty$ for all $|x|<1$. For $q<\varepsilon<1$ we have

$$-\int_{\varepsilon}^{1}\frac{h(x)}{u(x)}\,\mathrm{d}x=-\int_{\varepsilon}^{1}\frac{\mathrm{d}x}{(x-c)(1-x)^m}+\int_{\varepsilon}^{1}\frac{\mathrm{d}x}{(x-c)(1-x)^{m+\beta}}.$$

The first integral on the right-hand side of this equality is certainly convergent as $0<m<1$, whilst the second is convergent or divergent depending on whether $m+\beta<1$ or $m+\beta\geqslant 1$. Thus whether condition (iv) of Theorem 1 is satisfied depends on whether $m+\beta<1$. Hence we have proved that if $m+\beta<1$ then there exists a MBPII, and the honest MBPII is unique and positive recurrent. Conversely, if $m+\beta\geqslant 1$ then no MBPII exists.

Note how simple this proof based on Theorem 1 is; an algebraic proof based on the original Proposition 1 would be much more demanding.

Example 2. Let us retain the $\{b_n\}$ of (4.1) but now consider the simpler immigration structure $\alpha_n=1/n$ $(n\geqslant 1)$. Here $h(x)=-\ln(1-x)<\infty$ for all $|x|<1$, with $h(1)=\sum_n\alpha_n=\infty$. We claim that for any $c<y<1$

$$-\int_{y}^{1}\frac{h(x)}{u(x)}\,\mathrm{d}x=\int_{y}^{1}\frac{\ln(1-x)}{(c-x)(1-x)^m}\,\mathrm{d}x<\infty. \tag{4.2}$$

Since $0<m<1$, we can find an r such that $0<r$ and $m+r<1$. Now

$$-\int_{y}^{1}\frac{(1-x)^{-r}}{(c-x)(1-x)^m}\,\mathrm{d}x<\infty, \tag{4.3}$$

and $\lim_{x\to 1}[\ln(1-x)/(1-x)^{-r}]=0$, so as integral (4.3) converges so must (4.2). Thus there exists a MBPII, and the honest MBPII is unique and positive recurrent.

To illustrate a different functional structure, let $\{\alpha_n\}$ and $\{b_n\}$ now involve the Riemann zeta forms $\alpha_n = \alpha n^{-\delta}$ $(n\geqslant 1)$ and $b_n = bn^{-v}$ $(n > 1)$, with $b_0 = b$ and $b_1 = -b\sum_{n=1}^{\infty} n^{-v}$, where α, $b > 0$. Then since we require $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n\neq 1} b_n < \infty$, we must have $\delta\leqslant 1$ and $v > 1$. Moreover, by Theorem 1 we need $u'(1) = \infty$ and thus $v\leqslant 2$, so we should only consider $1 < v\leqslant 2$. Even so, this situation still separates into four different cases, namely: $v = 2$ (Case 1) and $1 < v < 2$ with $\delta\leqslant 0$ (Case 2), $\delta = 1$ (Case 3) and $0 < \delta < 1$ (Case 4). Applying Theorem 1 immediately shows that a MBPII does not exist in Cases 1 and 2. Case 4 is the most interesting and so we shall examine this first. Here we use $f(x) \asymp g(x)$ to denote $\lim_{x\to 1} f(x)/g(x) = k$ for some constant $k > 0$.

Example 3. *Case* 4 $(0 < \delta < 1,\ 1 < v < 2)$: By an Abelian theorem we see that as $s \uparrow 1$, $\sum_{n=1}^{\infty} s^n/n^{\delta} \asymp (1 - s)^{\delta-1}$, whence it immediately follows that $h(s) \asymp (1 - s)^{\delta-1}$ and $u'(s) \asymp (1 - s)^{v-2}$. So the Mean Value Theorem yields $-h(s)/u(s) \asymp (1 - s)^{\delta-v}$. Hence whether $\int_{\varepsilon}^{1} -[h(s)/u(s)]\,\mathrm{d}s$ converges or diverges depends on whether $v-\delta < 1$ or $v - \delta\geqslant 1$. Applying Theorem 1 therefore shows that under this parameter regime with $0 < \delta < 1$ and $1 < v < 2$: if $v < 1 + \delta$ then there exists a MBPII and the honest MBPII is unique and positive recurrent, whilst if $v\geqslant 1+\delta$ then there exists no MBPII.

Pakes (1993) also considers this example, and concludes that when $v < 1 + \delta$ the MBPII is positive recurrent, and when $v\geqslant 1 + \delta$ it is null recurrent. Whilst the first conclusion agrees with ours, the case of null recurrence does not actually exist. This highlights an additional benefit of using Theorem 1, since it provides an automatic check on existence and so there is no need to provide a separate verification.

Example 4. *Case* 3 $(\delta=1,\ 1 < v < 2)$: Retain $h(s)$ as the p.g.f. of $\{\alpha_n\}$ with $0 < \delta^* < 1$, and let $h_1(s)$ denote the p.g.f. when $\delta = 1$, i.e. $h_1(s) = -\alpha\ln(1 - s)$. Then on applying l'Hôpital's rule,

$$\lim_{s\uparrow 1}\left[\frac{h_1(s)/u(s)}{h(s)/u(s)}\right] = \lim_{s\uparrow 1}\left[\frac{-\alpha\ln(1 - s)}{(1 - s)^{\delta^*-1}}\right] = \lim_{s\uparrow 1}\left[\left(\frac{\alpha}{1 - \delta^*}\right)(1 - s)^{1-\delta^*}\right] = 0.$$

Now for any given $1 < v < 2$, we can always choose an appropriate $0 < \delta^* < 1$ by taking $\delta^* > v - 1$, and as we have already seen that $\int_{\varepsilon}^{1} -[h(s)/u(s)]\,\mathrm{d}s$ converges it therefore follows that $\int_{\varepsilon}^{1} -[h_1(s)/u(s)]\,\mathrm{d}s$ converges. Thus there exists a MBPII and the honest MBPII is unique.

Case 2 is simple to prove, since $-h(s)/u(s) \asymp s(1 - s)^{-v}$, whilst Case 1 is trivial. Note that our conclusion that no MBPII exists for the above example when $\delta\leqslant 0$ also follows directly from Corollary 5.4 of Chen and Renshaw (1990) (see also Remark 5). For this result states that for an MBPII to exist we require $\sum_{n=1}^{\infty} \alpha_n/n < \infty$; in particular, we must have $\lim_{n\to\infty} \inf(\alpha_n) = 0$. Neither of these conditions is satisfied by $\alpha_n = \alpha n^{-\delta}$ when $\delta\leqslant 0$.

Finally, it must be stressed that there exists a much wider range of possibilities than the four examples considered here. For example, in the above Riemann zeta illustration, the two constants α and b can be replaced by two slowly varying functions. Specifically, let $\alpha_n = n^{-\delta}L_1(n)$ $(n \geqslant 1)$ and $b_n = n^{-v}/L_2(n)$ $(n > 1)$ where $L_1(n)$ and $L_2(n)$ are two non-zero slowly varying functions at infinity. Then (as kindly pointed out by the referee) the subsequent analysis develops in a similar manner to the above. In particular, for the most interesting case of $0 < \delta < 1$, $1 < v < 2$ we can easily prove that if $v < 1+\delta$ then there exists a MBPII, and the honest MBPII is unique and positive recurrent, whilst if $v > 1+\delta$ there exists no MBPII. The case of $v = 1+\delta$ is more delicate, since the outcome depends on the specific forms taken by $L_1(n)$ and $L_2(n)$.

5. Results for general $\{a_i\}$

Let us now return to the development of existence and recurrence results for the more general MBPII with rates (2.5)–(2.7). First note that the q-matrix Q of a general MBPII is a conservative uni-instantaneous (CUI) q-matrix, and so the results of Chen and Renshaw (1993b) are immediately applicable. We follow Chen and Renshaw (1990) by associating with Q two other q-matrices $Q^* = \{q^*_{ij}\}$ and $\tilde{Q} = \{\tilde{q}_{ij}\}$ over $E = \{0, 1, \ldots\}$ and $N = E \setminus \{0\}$, namely

$$\tilde{q}_{ij} = q_{ij} \quad \text{for } i,j \in N \tag{5.1}$$

and

$$q^*_{ij} = \begin{cases} q_{ij} & \text{if } i,j \in N, \\ b_0 & \text{if } i = 1,\ j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Since both $\tilde{Q}$ and Q^* are stable q-matrices, we can work in terms of $\tilde{\Phi}(\lambda) = \{\tilde{\phi}_{ij}(\lambda);\ i,j \in N\}$ and $\Phi^*(\lambda) = \{\phi^*_{ij}(\lambda);\ i,j \in E\}$, namely the Feller minimal resolvents of $\tilde{Q}$ and Q^*, respectively.

Theorem 2. *For any given q-matrix Q defined through* (2.5)–(2.7):

(i) *there exists a general MBPII if and only if for some* (*and therefore for all*) $\lambda > 0$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \alpha_j \phi^*_{jk}(\lambda) < \infty, \tag{5.3}$$

(ii) *when condition* (5.3) *is satisfied the honest general MBPII is unique, and the associated resolvent $R(\lambda) = \{r_{ij}(\lambda);\ i,j \geqslant 0\}$ is given by*

$$r_{ij}(\lambda) = \begin{cases} \rho(\lambda) & \text{if } i = j = 0, \\ \rho(\lambda)\eta_j(\lambda) & \text{if } i = 0,\ j \geqslant 1, \\ \rho(\lambda) z_i(\lambda) & \text{if } i \geqslant 1,\ j = 0, \\ \phi^*_{ij}(\lambda) + z_i(\lambda)\rho(\lambda)\eta_j(\lambda) & \text{if } i,j \geqslant 1, \end{cases}$$

where for $i,j\geqslant 1$

$$z_i(\lambda)=1-\lambda\sum_{j=1}^{\infty}\phi^*_{ij}(\lambda),\qquad \eta_j(\lambda)=\sum_{k=1}^{\infty}\alpha_k\phi^*_{kj}(\lambda)$$

and

$$\rho(\lambda)=\left[\lambda+\sum_{k=1}^{\infty}\alpha_k(1-z_k(\lambda))\right]^{-1}.$$

Proof. This follows directly by combining Theorem 6.1 of Chen and Renshaw (1993b) with the fact that $\tilde{\Phi}(\lambda)$ and $\Phi^*(\lambda)$ coincide in N (since zero is an absorbing state for the Q^*-process). The proof of uniqueness for the honest general MBPII directly parallels that of Chen and Renshaw (1990) for the ordinary MBPII. □

In Theorem 2, both the existence condition and the construction technique depend upon the Q^*-resolvent, which in practice will not be easy to handle. For example, it is unlikely that we could use Theorem 2 directly to determine whether a given process is positive recurrent. Fortunately, we can bypass this by applying Theorem 1 in conjunction with a simple lemma. Let ${}^{(1)}Q=\{{}^{(1)}q_{ij};\ i,j\in E\}$ and ${}^{(2)}Q=\{{}^{(2)}q_{ij};\ i,j\in E\}$ be two stable q-matrices, and denote ${}^{(1)}Q\leqslant{}^{(2)}Q$ if their individual components satisfy this inequality, i.e.

$${}^{(1)}q_{ij}\leqslant{}^{(2)}q_{ij}\quad(\forall i\neq j)\tag{5.4}$$

and

$${}^{(2)}q_i\equiv-{}^{(2)}q_{ii}\leqslant-{}^{(1)}q_{ii}\equiv{}^{(1)}q_i\quad(\forall i).\tag{5.5}$$

Note that neither ${}^{(1)}Q$ nor ${}^{(2)}Q$ need be conservative; indeed if they are we have the uninteresting situation of ${}^{(1)}Q={}^{(2)}Q$. Similarly, the two resolvents ${}^{(1)}\Phi(\lambda)\leqslant{}^{(2)}\Phi(\lambda)$ iff this inequality holds for all $\lambda>0$.

Lemma 3 (Comparison theorem)**.** *Let* ${}^{(1)}Q$ *and* ${}^{(2)}Q$ *be two stable q-matrices with associated Feller minimal resolvents* ${}^{(1)}\Phi(\lambda)$ *and* ${}^{(2)}\Phi(\lambda)$, *respectively. Then*

$${}^{(1)}Q\leqslant{}^{(2)}Q\quad\text{if and only if}\quad{}^{(1)}\Phi(\lambda)\leqslant{}^{(2)}\Phi(\lambda).\tag{5.6}$$

Similarly, on denoting ${}^{(1)}F(t)$ *and* ${}^{(2)}F(t)$ *to be the associated Feller minimal transition functions,*

$${}^{(1)}Q\leqslant{}^{(2)}Q\quad\text{if and only if}\quad{}^{(1)}F(t)\leqslant{}^{(2)}F(t).\tag{5.7}$$

Proof. First note that $\lim_{\lambda\to\infty}\lambda(\lambda\Phi(\lambda)-I)=Q$. Then for all $\lambda>0$, if ${}^{(1)}\Phi(\lambda)\leqslant{}^{(2)}\Phi(\lambda)$, it follows that $\lambda(\lambda{}^{(1)}\Phi(\lambda)-I)\leqslant\lambda(\lambda{}^{(2)}\Phi(\lambda)-I)$, whence on letting $\lambda\to\infty$ we immediately obtain ${}^{(1)}Q\leqslant{}^{(2)}Q$. Conversely, assume that ${}^{(1)}Q\leqslant{}^{(2)}Q$. Then following Feller (1940) (see also Yang, 1990) we know that the Feller minimal resolvent for any given q-matrix may be obtained from the following iterative procedure. For every $\lambda>0$ and $i,j\in E$ let

$$\phi^{(1)}_{ij}(\lambda)=\frac{\delta_{ij}}{\lambda+q_i}\quad\text{and}\quad\phi^{(n+1)}_{ij}(\lambda)=\frac{\delta_{ij}}{\lambda+q_i}+\sum_{k\neq i}\frac{q_{ik}}{\lambda+q_i}\phi^{(n)}_{kj}(\lambda).\tag{5.8}$$

Then $\phi_{ij}^{(n)}(\lambda) \uparrow \phi_{ij}(\lambda)$ as $n \to \infty$. Now it is clear from (5.5) that

$$ {}^{(1)}\phi_{ij}^{(1)}(\lambda) = \frac{\delta_{ij}}{\lambda + {}^{(1)}q_i} \leqslant \frac{\delta_{ij}}{\lambda + {}^{(2)}q_i} = {}^{(2)}\phi_{ij}^{(1)}(\lambda). \tag{5.9} $$

Whence on using (5.4), (5.5) and (5.9), applying the induction principle to expression (5.8) then shows that for all $n \geqslant 1$, $i, j \in E$ and $\lambda > 0$

$$ {}^{(1)}\phi_{ij}^{(n)}(\lambda) \leqslant {}^{(2)}\phi_{ij}^{(n)}(\lambda). \tag{5.10} $$

Thus

$$ {}^{(1)}\phi_{ij}(\lambda) \leqslant {}^{(2)}\phi_{ij}(\lambda) \quad (\forall i, j \in E;\ \forall \lambda > 0), $$

which concludes the proof of (5.6). The proof of (5.7) follows similarly since the Feller minimal transition function obeys a similar iteration procedure (see, for example, Yang, 1990). □

We shall now apply Lemma 3 to our general MBPII (denoted by ${}^{(g)}q_{ij}$, etc.) by comparing it to the ordinary MBPII (denoted by ${}^{(o)}q_{ij}$, etc.) for which $a_i \equiv 0$; this is particularly instructive since far more is known about the latter than the former. Let ${}^{(\cdot)}\phi_{ij}^{*}(\lambda)$ and ${}^{(\cdot)}f_{ij}^{*}(t)$ denote the elements of ${}^{(\cdot)}\Phi^{*}(\lambda)$ and ${}^{(\cdot)}F^{*}(t)$, respectively.

Lemma 4. *For general rates* $\{a_i;\ i \geqslant 1\}$ *we have for every* $i, j \geqslant 1$ *and* $\lambda > 0$:

(i)

$$ {}^{(g)}\phi_{ij}^{*}(\lambda) \leqslant {}^{(o)}\phi_{ij}^{*}(\lambda) \quad \textit{and} \quad {}^{(g)}f_{ij}^{*}(t) \leqslant {}^{(o)}f_{ij}^{*}(t); \tag{5.11} $$

(ii) *if* $a_i \equiv a$ *for all* $i \geqslant 1$, *then*

$$ {}^{(g)}\phi_{ij}^{*}(\lambda) = {}^{(o)}\phi_{ij}^{*}(\lambda + a) \quad \textit{and} \quad {}^{(g)}f_{ij}^{*}(t) = {}^{(o)}f_{ij}^{*}(t)\mathrm{e}^{-at}. \tag{5.12} $$

Proof. (i) is a direct consequence of Lemma 3, whilst the proof of (ii) is straightforward. □

We have shown earlier that the conditions of Theorem 1 are simple to check for the restricted MBPII (with $a_i = ai = 0$), and we shall now use Lemma 4 to show that these same conditions are also sufficient for any general MBPII.

Theorem 5. *For any q-matrix Q defined through* (2.5)–(2.7), *if the conditions in Theorem* 1 *are satisfied then there exists a general MBPII.*

Proof. This follows as a direct consequence of (1.1), (5.3) and (5.11). □

If $\{a_i\}$ is bounded, which is the usual situation in practice, then these conditions are necessary as well as sufficient.

Theorem 6. *Suppose* $\{a_i\}$ *is bounded for a given q-matrix Q defined through* (2.5)–(2.7). *Then there exists a general MBPII if and only if the conditions in Theorem* 1 *are satisfied.*

Proof. Given Theorem 5, we clearly only need prove necessity. First note that for the special case of $a_i \equiv a > 0$ for all $i \geqslant 1$, the result follows directly from (5.12). Whilst for the general case in which a_i is not constant, since $\{a_i\}$ is bounded denote $M = \sup_{i\geqslant 1}\{a_i\} < \infty$. Now define ${}^{(M)}Q$ to be the q-matrix as given by (2.5)–(2.7), which corresponds to the special case $a_i \equiv M > 0$ for all $i \geqslant 1$. The associated rates (5.2) for ${}^{(M)}Q^*$ then become

$$ {}^{(M)}q^*_{ij} = \begin{cases} ib_{j-i} & \text{if } j \geqslant i-1,\ j \neq i,\ i \geqslant 1, \\ ib_1 - M & \text{if } j = i,\ i \geqslant 1, \\ 0 & \text{otherwise.} \end{cases} $$

It is easy to see that ${}^{(M)}Q^*$ is a stable q-matrix, and that

$$ {}^{(M)}Q^* \leqslant {}^{(\mathrm{g})}Q^* \leqslant {}^{(\mathrm{o})}Q^*. $$

Thus as ${}^{(M)}Q^*$, ${}^{(\mathrm{g})}Q^*$ and ${}^{(\mathrm{o})}Q^*$ are all stable q-matrices, Lemma 3 implies that

$$ \alpha^{(M)}\Phi^*(\lambda) \leqslant \alpha^{(\mathrm{g})}\Phi^*(\lambda) \leqslant \alpha^{(\mathrm{o})}\Phi^*(\lambda). \tag{5.13} $$

The result now follows on applying (1.1) and the inequality (5.13). □

Although we have proved that when $\{a_i\}$ is bounded, the sufficiency condition in Theorem 5 is also necessary, it is not reasonable to expect this to apply to the general case where $\{a_i\}$ is not bounded. Indeed, when $a_i = ia > 0$, Chen and Renshaw (1995) prove that there exists a general MBPII (called a BLUE process) if and only if $\sum_{n=1}^{\infty}(\alpha_n/n) < \infty$. Thus our conditions in Theorem 1 are not necessary in this case. This situation can be generalised as follows.

Theorem 7. *Suppose that Q satisfies the condition that $\inf_{i\geqslant 1}(a_i/i) > 0$. Then if $\sum_{n=1}^{\infty}(\alpha_n/n) < \infty$, there exists a general MBPII. Moreover, if Q satisfies the further condition that $0 < \inf_{i\geqslant 1}(a_i/i) \leqslant \sup_{i\geqslant 1}(a_i/i) < \infty$, then there exists a general MBPII with this given Q if and only if $\sum_{n=1}^{\infty}(\alpha_n/n) < \infty$.*

Proof. This follows easily on using Lemma 3. □

We may continue along this route to obtain yet more results. For example, it is not difficult to develop an exact existence condition for the case of $a_n = n^k$ for fixed integer $k > 0$. This case can then be generalised to $b_1 n^k \leqslant a_n \leqslant b_2 n^k$ $(n = 1, 2, \ldots)$ for $b_2 > b_1 > 0$. However, rather than considering such easy generalisations, let us return to the recurrence property of general MBPII.

Theorem 8. *Any general MBPII is not only recurrent, it is also positive recurrent.*

Proof. We first note that every general MBPII is irreducible, and so it is positive recurrent for all states if and only if it is positive recurrent for a given state. Thus if we denote the resolvent of the general MBPII by ${}^{(\mathrm{g})}R(\lambda) = \{{}^{(\mathrm{g})}r_{ij}(\lambda); i, j \geqslant 0, \lambda > 0\}$, then it is positive recurrent if and only if

$$ \lim_{\lambda\to 0} \lambda\, {}^{(\mathrm{g})}r_{00}(\lambda) > 0. \tag{5.14} $$

Now from part (ii) of Theorem 2 we know that

$$^{(g)}r_{00}(\lambda) = \left[\lambda + \lambda \sum_{k=1}^{\infty} \alpha_k \sum_{j=1}^{\infty} {}^{(g)}\phi^*_{kj}(\lambda)\right]^{-1}.$$

Whilst from Lemma 3 we have

$$^{(g)}\Phi^*(\lambda) \leqslant {}^{(o)}\Phi^*(\lambda),$$

and so

$$^{(g)}r_{00}(\lambda) \geqslant \left[\lambda + \lambda \sum_{k=1}^{\infty} \alpha_k \sum_{j=1}^{\infty} {}^{(o)}\phi^*_{kj}(\lambda)\right]^{-1} = {}^{(o)}r_{00}(\lambda) \quad (\forall \lambda > 0). \tag{5.15}$$

However, we have proved that for ordinary MBPII, i.e. with $a_i \equiv 0$, the process is positive recurrent (see Theorem 1), and so

$$\lim_{\lambda \to 0} \lambda\, {}^{(o)}r_{00}(\lambda) > 0. \tag{5.16}$$

Combining (5.15) and (5.16) yields (5.14), which concludes the proof. □

6. Equilibrium distributions

From Theorem 8 we see that any general MBPII is positive recurrent and irreducible. The equilibrium distribution must therefore always exist, and so to conclude we shall determine its structure. First note that it is easy to show that the inverse functions of $\sigma(t)$ and $q(t)$ (defined in Section 3) possess the same integrand; only the limits of integration differ. Let us therefore consider $\xi(s)$ to be any function having the property that $\xi'(s) = 1/u(s)$. Then the two inverse functions take the forms

$$\sigma^{-1}(s) = \xi(s) - \xi(1) \quad \text{if } q < s \leqslant 1,$$
$$q^{-1}(s) = \xi(s) - \xi(0) \quad \text{if } 0 \leqslant s < q$$

which suggests taking the function

$$\xi(s) \equiv \int \frac{\mathrm{d}s}{u(s)} = \begin{cases} q^{-1}(s) & \text{if } 0 \leqslant s < q, \\ \sigma^{-1}(s) & \text{if } q < s \leqslant 1. \end{cases} \tag{6.1}$$

Let us consider the general class of MBPII for which $a_i \equiv a > 0$, that is the population is subject to an externally induced catastrophe. Note that the equilibrium distribution for $a = 0$ has already been obtained by Pakes (1993), and so is not considered here.

Theorem 9. *The equilibrium distribution* $\{\pi_j; j \geqslant 0\}$ *of a general MBPII with* $a_i \equiv a > 0$ *has generating function*

$$\Gamma(s) \equiv \sum_{j=0}^{\infty} \pi_j s^j = \pi_0 \left[1 + \int_0^s \frac{h(q) - h(y)}{u(y)} \exp\{-a\xi(y)\}\, \mathrm{d}y\right] \tag{6.2}$$

with

$$\pi_0 = \left(1 + \int_0^1 \frac{h(q) - h(s)}{u(s)} \exp\{-a\xi(s)\}\,\mathrm{d}s\right)^{-1}, \tag{6.3}$$

where $h(s)$ *and* $u(s)$ *are given in* (1.3) *and* $\xi(s)$ *is defined through* (6.1).

Proof. On applying Theorem 2 and Lemma 4 to this process, it can easily be shown that for $j \geqslant 1$

$$\pi_j = \pi_0 \sum_{k=1}^{\infty} \alpha_k \phi_{kj}^*(a) \quad \text{where } \pi_0 = \left(1 + \sum_{k=1}^{\infty} \alpha_k \sum_{j=1}^{\infty} \phi_{ij}^*(a)\right)^{-1} \tag{6.4}$$

and $\{\phi_{ij}^*(\lambda)\}$ is the Q_1-resolvent for the branching process without immigration. Note that the existence conditions guarantee that the right-hand sides of (6.4) are well-defined and finite. Use of the branching property then effects the further reduction

$$\pi_0 = \left(1 + \int_0^{\infty} \mathrm{e}^{-at}[h(\sigma(t)) - h(q(t))]\,\mathrm{d}t\right)^{-1}.$$

On noting that $\sigma(t) \downarrow \sigma = q \uparrow q(t)$ as $t \to \infty$, using the fact that both $\sigma(t)$ and $q(t)$ satisfy the same differential equation $\mathrm{d}x(t)/\mathrm{d}t = u(x(t))$ then leads to

$$\pi_0 = \left(1 + \int_0^q \frac{h(q) - h(s)}{u(s)} \exp\left\{-a \int_0^s \frac{\mathrm{d}y}{u(y)}\right\} \mathrm{d}s \right.$$
$$\left. + \int_q^1 \frac{h(q) - h(s)}{u(s)} \exp\left\{-a \int_s^1 \frac{\mathrm{d}y}{u(y)}\right\} \mathrm{d}s\right)^{-1}.$$

Employing convention (6.1) then yields (6.3).

Moreover, using (6.4) we may write

$$\sum_{j=1}^{\infty} \pi_j s^j = \pi_0 \sum_{k=1}^{\infty} \alpha_k \left[\sum_{j=0}^{\infty} \phi_{kj}^*(a) s^j - \phi_{k0}^*(a)\right]$$
$$= \pi_0 \int_0^{\infty} \mathrm{e}^{-at}[h(F(s,t)) - h(q(t))]\,\mathrm{d}t,$$

where $F(s,t) = \sum_{j=0}^{\infty} p_{1j}^*(t) s^j$ is the generating function of transition functions of the underlying branching process without immigration. A little further algebra then results in (6.2). □

Note that if we place $a = 0$ in (6.2) and (6.3), then we can recover the equilibrium distribution for the MBPII with $a_i = 0$ given in Pakes (1993). In general it is unlikely that we could construct such simple closed-form solutions, though we can determine bounding properties. For example, suppose that the $\{a_i\}$ are bounded, i.e. there exist two constants $c_1 \geqslant 0$ and $c_2 > 0$ such that

$$0 \leqslant c_1 = \inf_{i \geqslant 1} a_i \leqslant \sup_{i \geqslant 1} a_i = c_2 < \infty. \tag{6.5}$$

Then the following result follows as a direct consequence of Lemma 3 and Theorem 9.

Theorem 10. *If the* $\{a_i;\ i\geqslant 1\}$ *satisfy* (6.5), *then there exists a general MBPII if and only if the conditions in Theorem* 1 *hold true, and under these conditions the honest MBPII is unique, irreducible and positive recurrent. Moreover, the generating function of the equilibrium distribution,* $\Gamma(s)$, *satisfies (for* $|s|\leqslant 1$) *the inequalities*

$$\left(1+\int_0^1 \frac{h(q)-h(s)}{f(s)}\exp\{-c_2\xi(s)\}\,\mathrm{d}s\right)^{-1} \leqslant \Gamma(0) \leqslant \left(1+\int_0^1 \frac{h(q)-h(s)}{f(s)}\exp\{-c_1\xi(s)\}\,\mathrm{d}s\right)^{-1} \tag{6.6}$$

and

$$1+\int_0^s \frac{h(q)}{f(y)}\exp\{-c_1\xi(y)\}\,\mathrm{d}y \leqslant \frac{\Gamma(s)}{\Gamma(0)} \leqslant 1+\int_0^s \frac{h(q)-h(y)}{f(y)}\exp\{-c_2\xi(y)\}\,\mathrm{d}y. \tag{6.7}$$

Finally, we present a specific example to illustrate the ease with which Theorem 9 can be applied. Consider a general MBPII with $\{\alpha_n\}$ and $\{b_n\}$ as given in Example 1 of Section 4, but with $a_i \equiv a\geqslant 0$. By applying Theorems 1 and 6 we can prove that when $m+\beta<1$ with $0<m<1$ and $0<\beta<1$, the honest MBPII is unique and possesses an equilibrium distribution. To obtain the generating function $\Gamma(s)$ via (6.2), we first need to evaluate integral (6.1) as

$$\xi(s)=\int \frac{\mathrm{d}s}{(c-x)(1-x)^m}.$$

This $\xi(s)$ is an elementary function if m is a rational number. For example, on taking $m=\frac{1}{2}$ we have

$$\xi(s)=\begin{cases} -[1/\sqrt{1-c}]\ln[\mu_1\eta(s)] & \text{if } 0\leqslant s<c,\\ -[1/\sqrt{1-c}]\ln[\eta(s)] & \text{if} c<s\leqslant 1,\end{cases}$$

where

$$\eta(s)=\frac{\sqrt{1-c}-\sqrt{1-s}}{\sqrt{1-c}+\sqrt{1-s}} \quad\text{and}\quad \mu_1=\frac{\sqrt{1-c}+1}{\sqrt{1-c}-1}.$$

Then $\Gamma(s)$ takes the form

$$\Gamma(s)=\pi_0\begin{cases} 1+\displaystyle\int_0^s \frac{(1-c)^{-\beta}-(1-x)^{-\beta}}{(c-x)\sqrt{1-x}}[\mu_1\eta(x)]^{a/\sqrt{1-c}}\,\mathrm{d}x & \text{if } 0\leqslant s<c,\\ 1+\lambda_1+\displaystyle\int_c^s \frac{(1-c)^{-\beta}-(1-x)^{-\beta}}{(c-x)\sqrt{1-x}}[\eta(x)]^{a/\sqrt{1-c}}\,\mathrm{d}x & \text{if } c<s\leqslant 1,\end{cases} \tag{6.8}$$

where $\pi_0=(1+\lambda_1+\lambda_2)^{-1}$ and

$$\lambda_1=\int_0^c \frac{(1-c)^{-\beta}-(1-x)^{-\beta}}{(c-x)\sqrt{1-x}}[\mu_1\eta(x)]^{a/\sqrt{1-c}}\,\mathrm{d}x,$$

$$\lambda_2=\int_c^1 \frac{(1-c)^{-\beta}-(1-x)^{-\beta}}{(c-x)\sqrt{1-x}}[\eta(x)]^{a/\sqrt{1-c}}\,\mathrm{d}x.$$

Acknowledgements

We wish to thank the referee for his detailed and constructive comments, which have substantially improved the paper.

References

Asmussen, S., Hering, H., 1983. Branching Processes. Birkhäuser, Boston.
Athreya, K.B., Jagers, P., 1996. Classical and Modern Branching Processes. Springer, Berlin.
Athreya, K.B., Ney, P.E., 1972. Branching Processes. Springer, Berlin.
Chen, A., Renshaw, E., 1990. Markov branching processes with instantaneous immigration. Probab. Theoret. Related Fields 87, 209–240.
Chen, A., Renshaw, E., 1993a. Recurrence of Markov branching processes with immigration. Stochastic Process. Appl. 45, 231–242.
Chen, A., Renshaw, E., 1993b. Existence and uniqueness criteria for conservative uni-instantaneous denumerable Markov processes. Probab. Theoret. Related Fields 94, 427–456.
Chen, A., Renshaw, E., 1995. Markov branching processes regulated by emigration and large immigration. Stochastic Proc. Appl. 57, 339–359.
Feller, W., 1940. On the integro-differential equations of purely discontinuous Markov processes. Trans. Amer. Math. Soc. 48, 488–515.
Foster, J.H., 1971. A limit theorem for a branching process with state-dependent immigration. Ann. Math. Statist. 42, 1773–1776.
Harris, T.E., 1963. The Theory of Branching Processes. Springer, Berlin.
Pakes, A.G., 1971. A branching process with a state-dependent immigration component. Adv. Appl. Probab. 3, 301–314.
Pakes, A.G., 1993. Absorbing Markov and branching processes with instantaneous resurrection. Stochastic Process. Appl. 48, 85–106.
Yamazato, M., 1975. Some results on continuous time branching processes with state-dependent immigration. J. Math. Soc. Japan 27, 479–496.
Yang, X.Q., 1990. The Construction Theory of Denumerable Markov Processes. Wiley, New York.