Dr Eric Renshaw (University of Edinburgh): The authors are to be congratulated for presenting such a stimulating array of theoretical and practical aspects of space–time modelling. Long-range memory processes are of particular importance, and insight into their behaviour can be obtained by considering spatial persistance through the interaction model (Bartlett, 1971, 1975)

$$X_{ij}(t + \mathrm{d}t) = X_{ij}(t)(1 + \lambda\,\mathrm{d}t) + \sum_{r,s=-\infty}^{\infty} a_{rs}\{X_{ij}(t) - X_{i+r,j-s}\}\,\mathrm{d}t + \mathrm{d}Z_{ij}(t) + o(\mathrm{d}t)$$

where $\{X_{ij}(t)\}$ is a lattice process on $i, j = \ldots, -1, 0, 1, \ldots$ and $\{\mathrm{d}Z_{ij}(t)\}$ denotes white noise. The associated spectrum (Renshaw, 1984) for equal $\{X_{ij}(0)\}$ is

$$f(\omega_1, \omega_2; t) = (\sigma^2/\psi)[\exp(\psi t) - 1]$$

where

$$\psi(\omega_1, \omega_2) = 2\left\{\lambda + \sum_{r,s=-\infty}^{\infty} a_{rs}[1 - \cos(r\omega_1 + s\omega_2)]\right\},$$

and this is especially useful for seeing how the form of the interaction weights $\{a_{rs}\}$ determines overall spatial structure.

For example, the one-dimensional case with nearest neighbour weights $a_{-1} = a_1 = \alpha$, $a_r = 0$ (otherwise), leads to $\psi = 2\lambda + 8\alpha\sin^2(\frac{1}{2}\omega)$. If $\alpha < -\lambda/4$ then $\psi < 0$ for $\omega_0 < \omega \leqslant \pi$ where $\omega_0 = \cos^{-1}(1 + \lambda/2\alpha)$, whence $f(\omega; t) \sim -\sigma^2/\psi$ as $t \to \infty$. Thus ω_0 defines the 'outer scale of pattern' in the sense that if $\omega < \omega_0$ then $f(\omega; t)$ does not approach a stationary limit as $t \to \infty$. So changing λ enables us to alter the range of the scales of pattern present in the stationary part of the process. If $\lambda = 0$ then for small ω we have the inverse square law

$$f(\omega; t) \sim -(\sigma^2/2\alpha)\omega^{-2} \qquad (\alpha < 0),$$

while the Cauchy-type weights $a_r = k/[|r|(|r| + 1)]$ $(r \neq 0)$ yield pure '$1/\omega$ noise', i.e.

$$f(\omega; t) \sim (\sigma^2/2k\pi)\omega^{-1}.$$

Interest in spatial persistence requires us to extend this by constructing a process which possesses a genuine power law spectrum $f(\omega; t) \sim$ constant $\times\ \omega^{-d}$ for non-integer $d > 0$. This may be achieved by using a similar fractional differencing approach to the authors'. For the ARIMA(0, d, 0) process $x_t = (1 - B)^{-d}\varepsilon_t$ yields negative binomial weights which suggests putting

$$a_r = c\binom{|r| + d - 1}{|r|} \quad (r \neq 0).$$

These give rise to

$$\psi(\omega) = 2\lambda - 4c[2\sin(\tfrac{1}{2}\omega)]^d\cos[\tfrac{1}{2}(\pi - \omega)d],$$

and so

$$f(\omega; t) \sim [\sigma^2/4c\cos(\tfrac{1}{2}\pi d)]\omega^{-d} \qquad (\text{if } \lambda = 0)$$

as required.